STABILITY AND DYNAMICS OF LAMINATED SANDWICH PANELS WITH RANDOM MATERIAL PROPERTIES AND EXTERNAL LOADING

A *Thesis* Submitted
in partial Fulfillment of the Requirement for the
Degree of Master of Technology

By
DARSI NAGENDRA KUMAR

To
DEPARTMENT OF AEROSPACE ENGINEERING
INDIAN INSTITUTE OF TECHNOGY, KANPUR
February 2003

CERTIFICATE

It is certified that the work contained in this thesis entitled **"STABILITY AND DYNAMICS OF LAMINATED SANDWICH PANELS WITH RANDOM MATERIAL PROPERTIES AND EXTERNAL LOADING"**, by **Darsi Nagendra Kumar**, has been carried out under my supervision and that this work has not been submitted elsewhere for any degree.

Dayanand Yadav

February, 2003

Dr. Dayanand Yadav
Professor
Department of Aerospace Engineering
Indian Institute of Technology Kanpur
India.

ACKNOWLEDGEMENT

I feel great pleasure in expressing deep sense of gratitude and sincere thanks to my thesis supervisor **Dr. Dayanand Yadav** for his discerning guidance, invaluable suggestions and constant encouragement throughout the course of the thesis work. He taught me the values of research and independent thinking. Working with him, has been the most exciting phase of my career.

I am very much thankful to Prof N.G.R.Iyengar for introducing me to the world of *"Composite Materials"* that was completely new to me. I would like to thank Prof C.Venkatesan for introducing me the *"Aircraft Structural Analysis–1 & Theory of Vibrations"* and Prof C.S.Upadhyay for introducing me "Introduction to finite element methods".

I am highly thankful to Laxman, Mohite, Amit and Nazir sir for their valueble suggestions through out the thesis work.

I specially thank to all my friends Sridhar, Rajesh, Ravi, Srinu, Samson, Prasad, Sampath, K.S.V. Reddy, Anu, Sridevi and Divya for their help, inspiration, and love during stay at IIT Kanpur.

I appreciate and extend my thanks to all my lab mates Atul, Anil, Rishu, Hall-V & Hall-IV friends for their affection and care during stay at IIT Kanpur.

Darsi Nagendra Kumar

ABSTRACT

The use of composite laminates as structural members in various industries like aerospace, nuclear and automobile, has enormously increased due to their lightweight, high strength, high stiffness, excellent fatigue resistance and better thermal expansion coefficients. All engineering materials inherently show variations in material properties. Composite materials experience larger dispersions in their material properties compared to conventional (isotropic) materials. This is because of more number of parameters involved in their manufacturing/fabrication process. Change in factors such as fiber orientation, laminate thickness, volume fraction, curing temperature and pressure, voids, impurities, curing time etc. induce variations in the lamina properties. This leads to variation in stiffness coefficients of the laminates and thus results in uncertainty in the response behavior. The effect of variation can be accommodated in the formulation by modeling the material properties as random. The external loading is subject to variations due to uncertainties in the environment and is random in nature. Therefore, these variations in the stiffness parameters and external loading have to be identified in the probabilistic sense for accurate analysis of composite laminates. The influence of randomness in material properties and loading on buckling, natural frequencies and deflection behavior of laminated composite shells has been analyzed in the present study with perturbation technique. The basic formulation of the problem is developed based on Classical Laminate Theory.

Dedicated
to
My Loving *Parents*

LIST OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

A_{ij}	Extensional stiffness coefficients
a	X-dimension of the shell element
B_{ij}	Coupling stiffness coefficients
b	Y-dimension of the shell element
b_l	Basic random variable
D_{ij}	Bending stiffness coefficients
$\cdot^d$	A deterministic term
E []	Expectation of
E_{ij}	Elastic modulus of
F ( )	Function of
G_{ij}	Shear modulus
h	thickness of laminate
Q_{ij}	Stiffness coefficients
$\overline{Q}_{ij}$	Transformed stiffness coefficients
$\cdot^r$	Zero-mean random variable
RV	Random variable
SD	Standard deviation
th	Lamina thickness
u	Deflection along X-axis
Var ( )	Variance of
v	Deflection along Y-axis
w	Deflection along W-axis
ε	Small parameter
θ	Fiber orientation
υ_{ij}	Poisson's ratio

CHAPTER 1

INTRODUCTION

1.1 GENERAL INTRODUCTION

Composite materials are extensively used by various engineering industries in various structural applications. Now a days, aircrafts are built with a very high percentage of components made from composite materials. Composite materials have many advantages over conventional materials, because of their superior performance in terms of their strength to stiffness ratio. Composite laminates have the added advantage that their structural properties may be tailored according to the design requirements. This creates new possibilities and challenges for the analyst and the designer and makes accurate analysis a necessity for sensitive applications.

The mechanical and physical properties of these materials are dependent on the production and fabrication conditions. All materials have dispersion in material properties due to lack of complete control over manufacturing/fabrication/processing techniques generally employed. Composite materials experience larger dispersions in their material properties compared to conventional (isotropic) materials due to larger number of parameters involved in their fabrication. In addition, the uncertainties involved in manufacturing and processing techniques employed for composites are more

as compared to techniques used for isotropic materials. These uncertainties can occur due to air entrapment, delamination and lack of resin, incomplete curing of resin, excess resin between layers etc.

The conventional methods of design and anlysis of structures assumes the various parameters to be a constant. This approach ignores the variations in the system parameters and may not be acceptable for sensitive applications. The variations in the lamina properties result in variations in the stiffness coefficients of the laminates. Therefore, these variations in the stiffness parameters have to be identified in the probabilistic sense for accurate analysis of composite laminates.

The randomness occurs mainly in the following two groups of parameters:

- Geometric parameters like fibre orientation 'θ', and lamina thickness 'th'etc;
- Material property randomness like elastic modulus, Poisson's ratio, etc. of both fibre as well as the matrix.

Considering the above aspects, the properties of structural materials may be modelled as random variables.

In general, we may not have enough data about the conditions of the external loadings in engineering problems. In such cases, the exact estimation of these forces may not be feasible and it may only be possible to describe these as random. Therefore, the problem formulation with random external loading is more general, since it attempts to describe the system behaviour for whole class of possibilities rather than for a single one.

When geometric or material properties and external loading are random in nature, the derived response parameters like deflections, natural frequencies etc. are also random, being functions of these basic system parameters and random loading. Depending on the

characteristics of the input random variables, the statistics of the response parameter can vary.

In such conditions, statistical analysis not only provides an accurate method of analysis of the structural behaviour and determination of probability of failure for a given structure but also it can provide useful information on how to modify the structure to improve its performance.

SHELLS

The salient features of shells, as compared to other structural forms such as beams, frames and plates can be

- Efficiency of load carrying behaviour;
- High degree of reserved strength and structural integrity;
- High strength to weight ratio;
- Very small thickness ratio to other dimensions like span and radius of curvature;
- Very high stiffness;
- Containment of space.

The usage of fibre reinforced and laminated composite materials have enormously increased in the domain of application and range of structural efficiency of shell forms. Ground, as well as space vehicles, having shell forms, have been designed and successfully built of high strength temperature resistant composite materials. The skin of aircraft structures and ship hulls are composed of shell forms, built of stiffened shells with composite material bodies.

SANDWICH CONSTRUCTION

Instead of increasing the laminate thickness, sandwich construction is frequently used in aerospace applications. Sandwich with aluminium honeycomb cores or flex cores are used extensively in aerospace, land and water transportation and sports industries because of their high specific strength and stiffness. Some examples of structures composed of sandwich panels are launch vehicles and satelliteshells, helicopter rotor blades, aircraft control surfaces, ship hulls, jet engine nacelles and skin, high-speed trains, etc.

This type of construction consists of two thin facing layers separated by a core material. Several types of core shapes and core materials have been applied to the construction of sandwich structures. The core layer is made of low specific weight material like balsa, porous rubber, corrugated metal sheet, metallic and non-metallic honeycomb, etc., which may be much less stiff and strong than the face sheets.

In addition to the possibility of achieving high flexural-stiffness providing a smoother aerodynamic surface in a high-speed range, sandwich-type constructions also exhibit many properties of exceptional importance for aerospace and civil constructions. Among these are

- Excellent thermal and sound insulation;
- A longer time of exploitation as compared to stiffened-reinforced structures;
- Possibility of being designed as to meet very close thermal distortion tolerances such as those required for communication satellite antennas and reflectors.

The most popular core is the honeycomb construction that consists of very thin foils in the form of hexagonal cells perpendicular to the facings. Face sheets usually consist of aluminium or fibre-reinforced composite laminates. Face sheets are stiff and strong because they carry most of the loads, while lightweight core separates face sheets so that a higher bending stiffness of the composite panel can be achieved.

Regardless of the face sheet and core materials, the sandwich panel is considered a composite structure because of its inherent inhomogeneous and anisotropic nature. Even though the concept of sandwich construction is not very new, it has only been adopted for primary structure parts very recently. This is because there are a variety of problem areas to be investigated and overcome when the sandwich construction is applied to design of dynamically loaded structures. To enhance the attractiveness of sandwich construction, it is thus essential to have an understanding of the local strength and dynamic characteristics of individual sandwich panel/beam members.

Advanced sandwich-type constructions imply the presence of thick orthotropic core with bonded anisotropic face sheets that are treated as composite laminates. This arrangement presents an opportunity to tailor both the physical and mechanical properties of the faces by proper selection of laminate materials, their stacking sequence and fibre orientation. Suitable selection of fibre orientation and stacking sequence can result in substantial improvements of the buckling strength and the response behaviour to a variety of load conditions. The transverse shear elastic module of the core layer can also be optimised to enhance the overall response behaviour of the sandwich constructions. As expected, analytical modelling of sandwich-type panels is much more intricate than that of the usual laminated composite structure. In contrast to the regular laminated

composite structure, for which the assumptions are postulated for the structure as a whole, in the case of sandwich-type constructions the assumptions involving the core layer are different from those associated with the face sheets. More over, the analysis of sandwich panels featuring laminated face sheets is much more complicated than that with single layered faces. The complexity is due to the presence of three types of asymmetries resulting from the lay-up sequences in the face sheets, namely:

- Asymmetry with respect to the mid-surface of the face sheets, referred to as face asymmetry, inducing face bending-stretching coupling;
- Asymmetry with respect to the mid-surface of the core, refer to as global asymmetry, which induces global bending-stretching coupling;
- Presence of ply-angles between the principle axis of orthotropy of the face sheets materials and the geometric axes of the panel, inducing a structural coupling between stretching and shearing.

1.2 LITERATURE SURVEY

Any structural analysis problem has combinations of the following three major categories.

1) Analysis involving different types of material;
2) Analysis involving different types of structures shape wise;
3) Analysis involving different types of loading.

These lead to three types of problems with probabilistic modelling – with randomness in material properties, geometry and external loading.

Nigam and Narayanan [1], Crandall [2], Lin [3] have studied various class of problems related to external loading as random while the system properties and geometry are deterministic.

Ibrahim [4] has presented a review of structural dynamics problems with parameter uncertainties. Manohar and Ibrahim [5] have reviewed number of topics on various formulations and solution techniques for structural dynamic problems with parameter uncertainties beyond the work reported in Ibrahim [4]. Vaicities [6] has obtained results for free vibrational analysis of beams with mass and flexural rigidity as random variables. Chen and Saroka [7] have investigated the response of a multi-degree dynamic system with statistical properties to deterministic excitations. They have used the perturbation technique to solve the governing equations of motion. Vinson and Sierakowski [8] have explained the behaviour of structures composed of composite materials. Leissa and Martin [9] have studied the free vibrations and buckling of the composite plates with variable fibre spacing. Zhang and Chen [10] have analysed complex stochastic structures subjected to arbitrary deterministic excitation. Salim et al. [11] have applied the perturbation technique for the static analysis of composite plates with various boundary conditions with elastic modulus, poison's ratio as basic random variables. Also, Salim et al. [12] have investigated the effect of basic random parameters on the natural frequencies of rectangular composite plates using the perturbation technique. Singh et al. [13] have studied the initial buckling, natural frequency of cylindrical panel and composite plate with random material properties and obtained the second order statistics of response. Astill et al. [14] have examined the problem of impact loading of structures with random geometric and material properties. Elishakoff et al.

[15] have developed deterministic governing equations and boundary conditions for mean and covariance functions of displacement for statically determinate beam with stochastically varying stiffness but with deterministic loading. Adali et al. [16] have discussed the problem of maximization of critical buckling load of an angle-ply laminated plate, when he properties of the plate are known to be scattered about some mean value. Yadav and Verma [17,18] have investigated free vibration for cylindrical shells with random material properties. Bucher and Brenner [19] have presented a method for analyses of structural systems with random properties subjected to dynamic loading. Low [20] has developed a fundamental but reliable and comprehensive approach of obtaining the exact roots for the frequency equation of beam systems.

Jagadish [21] has studied the static response of graphite/epoxy and glass/epoxy composite plates with random material properties to random loading using the classical laminate theory. Onkar [22] has analysed the non-linear response behaviour of composite laminated plates with random material properties and random external loading.

Gibson and Ashby [23] have explained the behaviour of cellular solids structure and their properties. Thomson [24] has presented a high-order theory for the analysis of multi-layer sandwich panels. Paik et al. [25] have analysed the strength characteristic of aluminium sandwich panels with aluminium honeycomb core theoretically and experimentally. Muc and Zuchara [26] have presented the analysis of a thin-walled sandwich laminate composite face subject to axial compression. Becker [27] has analysed, in closed–form, the thickness effect of regular honeycomb core material. Glass et al [28] have analysed Graphite/epoxy honeycomb core sandwich permeability under mechanical loads. Marimuthu [29] has studied the free vibration analysis of cylindrical

honeycomb sandwich laminate with random material properties using finite element method. Vinson [34] has explained the behaviour of shells composed of isotropic and composite materials.

1.3 PRESENT WORK

The behaviour of composite laminates with both kinds of randomness, namely, random material properties and random external loading, has been studied for a limited class of problems. The sandwich composite construction with random material property and loading has not been fully addressed. In aerospace structures, such situation occurs very frequently in many applications. Some examples are aerodynamic, jet and rocket exhaust noise, acoustic loading on aircraft, rocket and spacecraft panels. The water pressure on ship and submarine panels during high-speed operation and at the event of underwater explosion also experience loadings that are random in nature. The acoustic pressure loading due to explosion and launching of rockets and firing of guns on vehicle and building panels are also random.

The aim of present study is to investigate the stability and dynamics of honeycomb-cored composite shells of various shapes like cylinder, cone and sphere and finding the second order response statistics. The material properties and external loading are treated as random in nature. In the present work, the stiffness properties of lamina such as E_{11}, E_{22}, υ_{12}, are considered as the basic material variables, and are assumed to be random. The study is used to infer the performance of a typical satellite payload fairing.

CHAPTER 2

FORMULATION

2.1 INTRODUCTION

This section presents a general approach for analysis of composite laminates involving random material properties and random loading. For the formulation of the problem the basic material properties like longitudinal modulus, transverse modulus, Poisson's ratio and the external loading are taken as random in nature and all other parameters are treated as deterministic.

2.2 CURVED COMPOSITE PANELS

Figure2.1a and 2.1b show the geometry along with the stress resultants for a shell element. Let (x, y, z) denote the orthogonal curvilinear coordinates (or shell coordinates) such that the x and y curves are lines of curvature on the mid-surface z=0, and z curves are straight lines perpendicular to the mid-surface. R_1 and R_2 denote the values of the principal radii of curvatures of the mid-surface. The lines of principal curvature coincide with the coordinate lines.

The shell under consideration is composed of a finite number of N orthotropic layers. z_k and z_{k-1} be the top and bottom z coordinates of the k^{th} lamina.

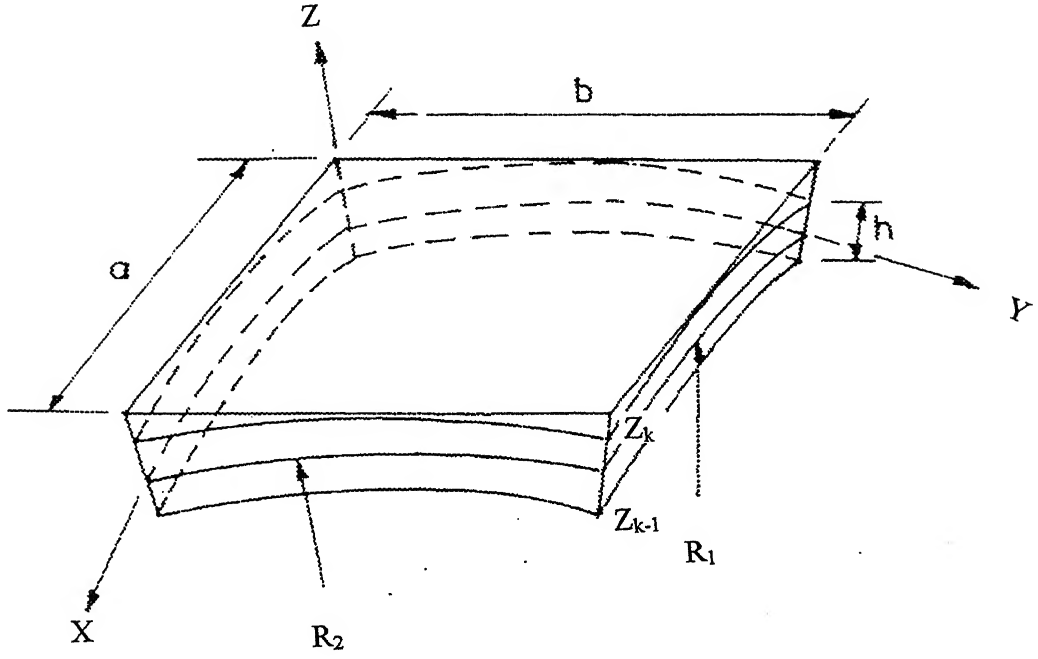


Figure 2.1a: Geometry of shell element

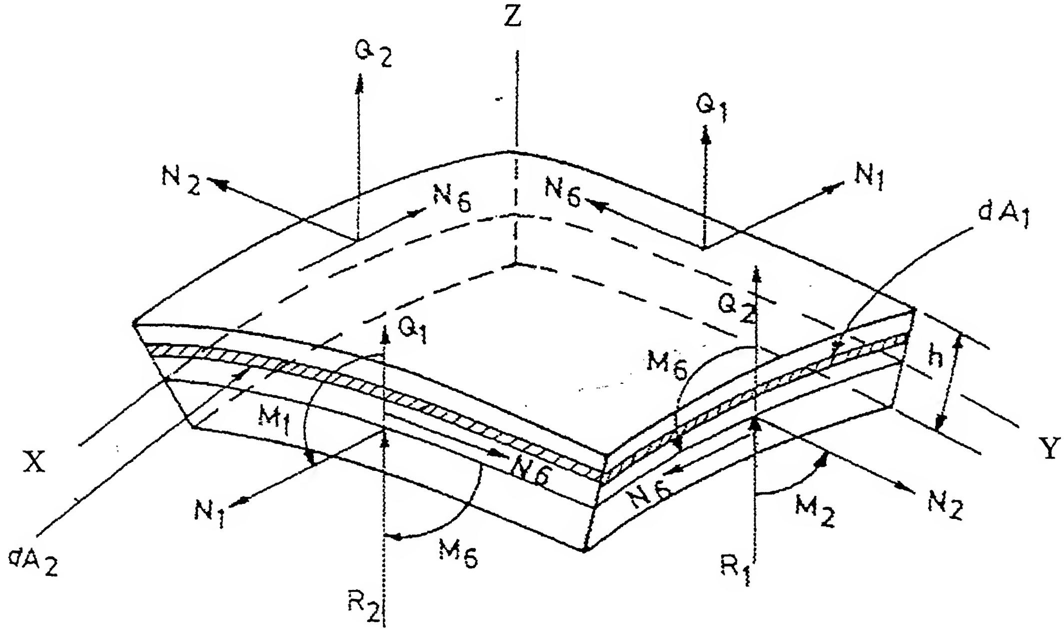


Figure2.1b: Geometry of shell element with stress resultants

2.2.1 THE DISPLACEMENT FIELD MODEL:

The displacement field relations for composite curved panels under consideration are [30]

$$\bar{u}(x, y, z, t) = (1+z/r_1)\, u + z.\, \phi_1 + z^2\, \varphi_1 + z^3\, \theta_1$$

$$\bar{v}(x, y, z, t) = (1+z/r_2)\, v + z.\, \phi_2 + z^2\, \varphi_2 + z^3\, \theta_2$$

$$\bar{w}(x, y, z, t) = w \tag{2.1}$$

where 't' is time, $\bar{u}$, $\bar{v}$ and $\bar{w}$ are displacements along the x, y and z coordinates respectively. u,v and w are the displacements of a point on the middle surface and ϕ_1 and ϕ_2 are the rotations at z=0 of normal to the mid-surface with respect to the y and x axes, respectively. Coefficients φ_1, φ_2, θ_1 and θ_2 have system dependent values.

Following the development by Shankara and Iyengar [31] for composite plates, a procedure is outlined here for curved panels. The displacement field is expressed using functions as coefficients

$$\bar{u} = (1+z/r_1)\, u + f_1(z).\phi_1 + f_2(z)\frac{\partial w}{\partial x}$$

$$\bar{v} = (1+z/r_2)\, v + f_1(z).\phi_2 + f_2(z)\frac{\partial w}{\partial y}$$

$$\bar{w} = w \tag{2.2}$$

The above equations can be written as

$$\bar{u} = (1+z/r_1)\, u + f_1(z).\phi_1 + f_2(z)\theta_1$$

$$\bar{v} = (1+z/r_2)\, v + f_1(z).\phi_2 + f_2(z)\theta_2$$

$$\bar{w} = w$$

where $\theta_1 = \dfrac{\partial w}{\partial x}$ and $\theta_2 = \dfrac{\partial w}{\partial y}$ (2.3)

The displacement vector for the model is, $\Lambda = (u, v, w, \theta_2, \theta_{1,} \phi_2, \phi_1)^T$

For Classical Laminate Theory, the displacement vector becomes $\Lambda = (u, v, w)^T$ (2.4)

2.2.2 STRAIN DISPLACEMENT RELATIONS

The strain displacement relations using equation (2.2) referred to Cartesian Coordinate system is

$$\varepsilon_1 = \varepsilon_1^0 + z(\kappa_1^0 + z^2\kappa_1^2) \qquad \varepsilon_4 = \varepsilon_4^0 + z^2\kappa_4^1$$

$$\varepsilon_2 = \varepsilon_2^0 + z(\kappa_2^0 + z^2\kappa_2^2) \qquad \varepsilon_5 = \varepsilon_5^0 + z^2\kappa_5^1$$

$$\varepsilon_6 = \varepsilon_6^0 + z(\kappa_6^0 + z^2\kappa_6^2) \tag{2.5}$$

where $\varepsilon_1^0 = \dfrac{\partial u}{\partial x} + \dfrac{w}{r_1}$; $\quad \varepsilon_2^0 = \dfrac{\partial v}{\partial y} + \dfrac{w}{r_2}$; $\quad \varepsilon_4^0 = \dfrac{\partial w}{\partial y} + \phi_2$;

$$\varepsilon_5^0 = \frac{\partial w}{\partial x} + \phi_1; \qquad \varepsilon_6^0 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y};$$

$$\kappa_1^0 = \frac{\partial \phi_1}{\partial x}; \qquad \kappa_2^0 = \frac{\partial \phi_2}{\partial y}; \qquad \kappa_6^0 = \frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_1}{\partial y};$$

$$\kappa_1^2 = \frac{-4}{3h^2}\left(\frac{\partial \phi_1}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right); \qquad \kappa_2^2 = \frac{-4}{3h^2}\left(\frac{\partial \phi_2}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right);$$

$$\kappa_6^2 = \frac{-4}{3h^2}\left(\frac{\partial \phi_2}{\partial x} + \frac{2\partial^2 w}{\partial x \partial y} + \frac{\partial \phi_1}{\partial y}\right); \qquad k_4{}^1 = \frac{-4}{h^2}\left(\phi_2 + \frac{\partial w}{\partial y}\right);$$

$$\kappa_5^1 = \frac{-4}{h^2}\left(\phi_1 + \frac{\partial w}{\partial x}\right) \tag{2.6}$$

Equation (2.5) gives the strain-displacement relations corresponding to the displacement field with mid-plane strains and curvatures.

2.2.3 STRESS-STRAIN RELATIONS

The stress-strain relations for k^{th} lamina in the material coordinate axis, whose fibers are oriented with respect to 'x' is given as [31]

$$\begin{Bmatrix} \sigma_l \\ \sigma_t \\ \sigma_{lt} \\ \sigma_{tz} \\ \sigma_{lz} \end{Bmatrix} = \begin{bmatrix} Q_{11}^K & Q_{12}^K & 0 & 0 & 0 \\ & Q_{22}^K & 0 & 0 & 0 \\ sym & & Q_{33}^K & 0 & 0 \\ & & & Q_{44}^K & \\ & & & & Q_{55}^K \end{bmatrix} \begin{Bmatrix} \varepsilon_l \\ \varepsilon_t \\ \varepsilon_{lt} \\ \varepsilon_{tz} \\ \varepsilon_{lz} \end{Bmatrix} \quad (2.7)$$

where 'l' and 't' denote the longitudinal and transverse directions of the lamina. σ_l, σ_t, ε_l, *and* ε_t are stresses and strains in the direction parallel and perpendicular to fiber direction. σ_{lt}, σ_{tz}, σ_{lz}, ε_{lz} and ε_{tz} are shear stresses and shear strains in the respective planes.

The stiffness coefficients are defined in terms of material properties as [32]

$$Q_{11} = \frac{E_{11}}{(1-\nu_{12}\nu_{21})} \qquad Q_{22} = \frac{E_{22}}{(1-\nu_{12}\nu_{21})} \qquad Q_{12} = \frac{E_{12}\nu_{22}}{(1-\nu_{12}\nu_{21})}$$

$$Q_{21} = Q_{12} \qquad Q_{66} = G_{12} \qquad Q_{44} = G_{13} \qquad Q_{55} = G_{23}$$

$$\nu_{12} = \nu_{21} \quad (2.8)$$

where E_{11} and E_{22} are longitudinal and transverse elastic moduli, G_{12} is in-plane shear modulus, G_{13}, G_{23} are the out-if-plane shear moduli and ν_{ij}'s are the Poisson's ratios.

The transformed stress-strain relations for the k[th] lamina with respect to the laminate coordinate system are [32]

$$\begin{Bmatrix} \sigma_1 \\ \sigma_{2t} \\ \sigma_6 \\ \sigma_4 \\ \sigma_5 \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11}^K & \overline{Q}_{12}^K & \overline{Q}_{16}^K & 0 & 0 \\ & \overline{Q}_{22}^K & \overline{Q}_{26}^K & 0 & 0 \\ & & \overline{Q}_{66}^K & 0 & 0 \\ sym & & & \overline{Q}_{44}^K & \overline{Q}_{45}^K \\ & & & & \overline{Q}_{55}^K \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix} \tag{2.9}$$

where $\overline{Q}_{ij}^K$ are the material constants of the k[th] lamina coordinate system [31,39].

2.2.4 EQUATIONS OF MOTION:

The equations of motion for curved panels subjected to in-plane loads, N_x, N_y, N_{xy} and distributed transverse dynamic loading 'q' including effects of transverse shear and rotary inertia may be written using the principle of virtual work [33].

$$\frac{\partial N_1}{\partial X}+\frac{\partial N_6}{\partial Y}=\overline{I}_1\ddot{u}+\overline{I}_2\ddot{\phi}_1-\gamma\overline{I}_3\frac{\partial\ddot{w}}{\partial X}$$

$$\frac{\partial N_6}{\partial X}+\frac{\partial N_2}{\partial Y}=\overline{I'}_1\ddot{v}+\overline{I'}_2\ddot{\phi}_2-\gamma\overline{I'}_3\frac{\partial\ddot{w}}{\partial Y}$$

$$\frac{\partial Q_1}{\partial X}+\frac{\partial Q_2}{\partial Y}-\gamma\frac{4}{h^2}\left(\frac{\partial K_1}{\partial X}+\frac{\partial K_2}{\partial Y}\right)+\gamma\frac{4}{3h^2}\left(\frac{\partial^2 P_1}{\partial X^2}+\frac{\partial^2 P_2}{\partial Y^2}+2\frac{\partial^2 P_6}{\partial X\partial Y}\right)-\frac{N_1}{R_1}-\frac{N_2}{R_2}+N_X\left(\frac{\partial^2 w}{\partial X^2}\right)+$$
$$N_Y\left(\frac{\partial^2 w}{\partial Y^2}\right)+N_{XY}\left(\frac{\partial^2 w}{\partial X\partial Y}\right)=\gamma\overline{I}_3\frac{\partial\ddot{u}}{\partial X}+\gamma\overline{I}_5\frac{\partial\ddot{\phi}_1}{\partial X}+\gamma\overline{I'}_3\frac{\partial\ddot{w}}{\partial Y}+\gamma\overline{I'}_5\frac{\partial\ddot{\phi}_2}{\partial Y}+I_1\ddot{w}-\gamma\frac{16}{9h^2}\left(\frac{\partial^2\ddot{w}}{\partial X^2}+\frac{\partial^2\ddot{w}}{\partial Y^2}\right)-q$$

$$\frac{\partial M_1}{\partial X}+\frac{\partial M_6}{\partial Y}-Q_1+\gamma\frac{4}{h^2}K_1-\gamma\frac{4}{3h^2}\left(\frac{\partial P_1}{\partial X}+\frac{\partial P_6}{\partial Y}\right)=I_2\ddot{u}+I_4\ddot{\phi}_1-\gamma\overline{I}_5\frac{\partial\ddot{w}}{\partial X}$$

$$\frac{\partial M_6}{\partial X}+\frac{\partial M_2}{\partial Y}-Q_2+\gamma\frac{4}{h^2}K_2-\gamma\frac{4}{3h^2}\left(\frac{\partial P_6}{\partial X}+\frac{\partial P_2}{\partial Y}\right)=\overline{I'_2}\ddot{v}+\overline{I'_4}\ddot{\phi}_2-\gamma\overline{I'_5}\frac{\partial\ddot{w}}{\partial Y} \tag{2.10}$$

where q (x, y, t) is the distributed transverse load and N_i, M_i etc are the stress resultants.

The Inertia's $\overline{I'_i}$ and $\overline{I}_i$, i=1,2,3,4 and 5 are defined by [30]

$$\overline{I}_1 = I_1 + \gamma \frac{2I_2}{R_1} \qquad \overline{I}_1 = I_1 + \gamma \frac{2I_2}{R_2}$$

$$\overline{I}_2 = I_2 + \frac{I_3}{R_1} - \gamma \frac{4I_4}{3h^2} - \gamma \frac{4I_5}{3h^2 R_1}$$

$$\overline{I'}_2 = I_2 + \frac{I_3}{R_2} - \gamma \frac{4I_4}{3h^2} - \gamma \frac{4I_5}{3h^2 R_2} \qquad I_3 = \frac{4I_4}{3h^2} - \frac{4I_5}{3h^2 R_1}$$

$$\overline{I'}_3 = \frac{4I_4}{3h^2} + \frac{4I_5}{3h^2 R_2} \qquad \overline{I}_4 = I_3 - \gamma \frac{8I_5}{3h^2} + \gamma \frac{16I_7}{9h^4}$$

$$\overline{I'}_4 = I_3 - \gamma \frac{8\overline{I}_5}{3h^2} + \gamma \frac{16I_7}{9h^4} \qquad \overline{I'}_5 = \frac{4I_5}{3h^2} - \frac{16I_7}{9h^4} \qquad (2.11)$$

$$(I_1, I_2, I_3, I_4, I_5, I_7) = \sum_{K=1}^{N} \int_{Z_K}^{Z_{K-1}} \rho^K (1, Z, Z^2, Z^3, Z^4, Z^6) dz$$

where, ρ^k is the mass per unit volume of the k^{th} lamina.

2.2.5 THE LAMINATE CONSTITUTIVE EQUATIONS:

It can be clearly seen from equation (2.9) that stresses in a composite laminate vary from layer to layer and hence a statically equivalent force and moment system is required for analysis. The stress resultants are expressed as (30)

$$(N_i, M_i, P_i) = \sum_{K=1}^{N} \int_{Z_K}^{Z_{K-1}} \sigma_i^K (1, Z, Z^3) dz \qquad i=1,2,6$$

$$(Q_1, K_1) = \sum_{K=1}^{N} \int_{Z_K}^{Z_{K-1}} \sigma_5^K (1, Z^2) dz$$

$$(Q_2, K_2) = \sum_{K=1}^{N} \int_{Z_K}^{Z_{K-1}} \sigma_4^K (1, Z^2) dz \qquad (2.12)$$

$$N_i = A_{ij}\varepsilon_j^0 + B_{ij}k_j^0 + \gamma E_{ij}k_j^2$$

$$M_i = B_{ij}\varepsilon_j^0 + D_{ij}k_j^0 + \gamma F_{ij}k_j^2$$

$$P_i = E_{ij}\varepsilon_j^0 + F_{ij}k_j^0 + H_{ij}k_j^2 \qquad i,j=1,2,6 \qquad (2.13)$$

Where, $A_{ij}, B_{ij}, D_{ij}, E_{ij}$, F_{ij} and H_{ij} are the laminate stiffness expressed as

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \sum_{K=1}^{N} \int_{Z_K}^{Z_{K-1}} \sigma^K (1, Z, Z^2, Z^3, Z^4, Z^6) dz \quad \text{i, j=1,2,6} \tag{2.14}$$

The equations for Classical Laminate Theory can be obtained by setting $\gamma = 0$.

Substituting all the above equations for curved panels in to the constitutive equations and then in to (2.10), we obtain equilibrium equations in terms of displacements. The equations of motion may now be written as

$$[L]\{\Lambda\} = \{q\} \tag{2.15}$$

where $\Lambda = (u, v, w)^T$ *and* $q = (0,0,q)^T$ for CLT and L is a matrix of differential operators. The above equations are valid for laminated composite curved panels.

2.3 CYLINDRICAL PANELS

2.3.1 EQUILIBRIUM EQUATIONS:

The general curved panel shown in fig 2.1a takes the form of a cylinder when $R_1=\infty$ and R_2 is constant. Using the geometrical definitions of cylindrical panels given in fig2.1a, equations (2.10) become

$$\frac{\partial N_1}{\partial X} + \frac{\partial N_6}{\partial Y} = \overline{I_1}\ddot{u}$$

$$\frac{\partial N_6}{\partial X} + \frac{\partial N_2}{\partial Y} = \overline{I'}_1\ddot{v}$$

$$\frac{\partial Q_1}{\partial X} + \frac{\partial Q_2}{\partial Y} - \frac{N_2}{R_2} + N_X\left(\frac{\partial^2 w}{\partial X^2}\right) + N_Y\left(\frac{\partial^2 w}{\partial Y^2}\right) + N_{XY}\left(\frac{\partial^2 w}{\partial X \partial Y}\right) = I_1\ddot{w} - q$$

where the modified inertias $\overline{I'_i}$, $\overline{I_i}$ are given as $\overline{I_1} = I_1$ and $\overline{I'_1} = I_1$

$$(I_1, I_2) = \sum_{K=1}^{N} \int_{Z_K}^{Z_{K-1}} \rho^K (1, Z) dz \tag{2.16}$$

2.3.2 STRAIN-DISPLACEMENT EQUATIONS:

The Strain–Displacement relations are obtained by using equations (2.5) and (2.6) and cylindrical panel geometry definitions (fig 1.1a):

$$\varepsilon_1^0 = \frac{\partial u}{\partial x} \qquad \varepsilon_2^0 = \frac{\partial v}{\partial y} + \frac{w}{r_2} \qquad \varepsilon_4^0 = \frac{\partial w}{\partial y}$$

$$\varepsilon_5^0 = \frac{\partial w}{\partial x} \qquad \varepsilon_6^0 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\kappa_1^0 = 0 \qquad \kappa_2^0 = 0 \qquad \kappa_6^0 = 0$$

$$\kappa_1^2 = \frac{-4}{3h^2}(\frac{\partial^2 w}{\partial x^2}) \qquad \kappa_2^2 = \frac{-4}{3h^2}(\frac{\partial^2 w}{\partial y^2}) \qquad \kappa_6^2 = \frac{-4}{3h^2}(\frac{2\partial^2 w}{\partial x \partial y})$$

$$\kappa_4^1 = \frac{-4}{h^2}(\frac{\partial w}{\partial y}) \qquad \kappa_5^1 = \frac{-4}{h^2}(\frac{\partial w}{\partial x}) \tag{2.17}$$

2.3.3 CONSTITUTIVE EQUATIONS:

The constitutive equations are obtained by using equations (2.13)

$$N_i = A_{ij}\varepsilon_j{}^0 \qquad M_i = 0 \qquad Q_i = 0 \qquad \text{i,j=1,2,6} \tag{2.18}$$

2.4 SPHERICAL PANELS:

2.4.1 EQUILIBRIUM EQUATIONS:

The general curved panel shown in fig 2.1a takes the form of a sphere when $R_1 = R_2 = R$.

Substituting the geometry definitions for spherical panels in to equation (2.10), we get

$$\frac{\partial N_1}{\partial X} + \frac{\partial N_6}{\partial Y} = \overline{I}_1 \ddot{u}$$

$$\frac{\partial N_6}{\partial X} + \frac{\partial N_2}{\partial Y} = \overline{I'}_1 \ddot{v}$$

$$\frac{\partial Q_1}{\partial X}+\frac{\partial Q_2}{\partial Y}-\frac{N_1}{R}-\frac{N_2}{R}+N_X(\frac{\partial^2 w}{\partial X^2})+N_Y(\frac{\partial^2 w}{\partial Y^2})+N_{XY}(\frac{\partial^2 w}{\partial X\partial Y})=I_1\ddot{w}-q$$

where the modified inertias $\overline{I'_i}$, $\overline{I_i}$ are given as $\overline{I_1}=I_1$ and $\overline{I'_1}=I_1$

$$(I_1,I_2)=\sum_{K=1}^{N}\int_{Z_K}^{Z_{K-1}}\rho^K(1,Z)dz \tag{2.19}$$

2.4.2 STRAIN-DISPLACEMENT EQUATIONS:

The Strain – Displacement relations are obtained by using equations (2.5) and (2.6) and spherical panel geometry definitions (Figure2.1a).

$$\varepsilon_1^0=\frac{\partial u}{\partial x}+\frac{w}{r_1} \qquad \varepsilon_2^0=\frac{\partial v}{\partial y}+\frac{w}{r_2} \qquad \varepsilon_4^0=\frac{\partial w}{\partial y}$$

$$\varepsilon_5^0=\frac{\partial w}{\partial x} \qquad \varepsilon_6^0=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}$$

$$\kappa_1^0=0 \qquad \kappa_2^0=0 \qquad \kappa_6^0=0$$

$$\kappa_1^2=\frac{-4}{3h^2}(\frac{\partial^2 w}{\partial x^2}) \qquad \kappa_2^2=\frac{-4}{3h^2}(\frac{\partial^2 w}{\partial y^2}) \qquad \kappa_6^2=\frac{-4}{3h^2}(\frac{2\partial^2 w}{\partial x\partial y})$$

$$\kappa_4^1=\frac{-4}{h^2}(\frac{\partial w}{\partial y}) \qquad \kappa_5^1=\frac{-4}{h^2}(\frac{\partial w}{\partial x}) \tag{2.20}$$

2.4.3 CONSTITUTIVE EQUATIONS:

The constitutive equations are obtained by using equations (2.13)

$$N_i=A_{ij}\varepsilon_j{}^0 \qquad M_i=0 \qquad Q_i=0 \qquad \text{i, j=1,2,6} \tag{2.21}$$

2.5 CONICAL PANELS:

2.5.1 EQUILIBRIUM EQUATIONS:

The general curved panel shown in fig 2.1a takes the form of a cone by setting R1= ∞ and R2 varying linearly along the x-axis gives a conical panel.

Substituting the geometry definitions for conical shells in to equation (2.10), we get

$$\frac{\partial N_1}{\partial X}+\frac{\partial N_6}{\partial Y}=\overline{I_1}\ddot{u}$$

$$\frac{\partial N_6}{\partial X}+\frac{\partial N_2}{\partial Y}=\overline{I'}_1\ddot{v}$$

$$\frac{\partial Q_1}{\partial X}+\frac{\partial Q_2}{\partial Y}-\frac{N_2}{R_2}+N_X(\frac{\partial^2 w}{\partial X^2})+N_Y(\frac{\partial^2 w}{\partial Y^2})+N_{XY}(\frac{\partial^2 w}{\partial X \partial Y})=I_1\ddot{w}-q$$

where the modified inertias $\overline{I'_i}$, $\overline{I_i}$ are given as $\overline{I_1}=I_1$ and $\overline{I'_1}=I_1$

$$(I_1, I_2)=\sum_{K=1}^{N}\int_{Z_K}^{Z_{K-1}}\rho^K(1,Z)dz \tag{2.22}$$

2.5.2 STRAIN-DISPLACEMENT EQUATIONS:

The Strain – Displacement relations are obtained by using equations (2.5) and (2.6) and conical panel geometry definitions.

$$\varepsilon_1^0=\frac{\partial u}{\partial x} \qquad \varepsilon_2^0=\frac{\partial v}{\partial y}+\frac{w}{r_2} \qquad \varepsilon_4^0=\frac{\partial w}{\partial y}$$

$$\varepsilon_5^0=\frac{\partial w}{\partial x} \qquad \varepsilon_6^0=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}$$

$$\kappa_1^0=0 \qquad \kappa_2^0=0 \qquad \kappa_6^0=0$$

$$\kappa_1^2=\frac{-4}{3h^2}(\frac{\partial^2 w}{\partial x^2}) \qquad \kappa_2^2=\frac{-4}{3h^2}(\frac{\partial^2 w}{\partial y^2}) \qquad \kappa_6^2=\frac{-4}{3h^2}(\frac{2\partial^2 w}{\partial x \partial y})$$

$$\kappa_4^1=\frac{-4}{h^2}(\frac{\partial w}{\partial y}) \qquad \kappa_5^1=\frac{-4}{h^2}(\frac{\partial w}{\partial x}) \tag{2.23}$$

2.5.3 CONSTITUTIVE EQUATIONS:

The constitutive equations are obtained by using equations (2.13)

$$N_i=A_{ij}\varepsilon_j{}^0 \qquad M_i=0 \qquad Q_i=0 \qquad \text{i, j=1,2,6} \tag{2.24}$$

2.6 GENERAL NATURE OF THE PROBLEM

The mathematical statement of a structural analysis problem usually involves three types of terms as follows.

1. Terms describing known system parameters.
2. Terms describing the known forcing function.
3. Terms describing the unknown response vector.

In general, the governing equations of any structural analysis problem can be expressed as:

$$[L]\{\Lambda\} = \{q\} \tag{2.25}$$

Where, $[L]$-System matrix (m X n), $\{\Lambda\}$-Response vector (n X 1) and $\{q\}$-Loading vector (m X 1).

In the above equation the characteristics of $[L]$ and $\{q\}$ are known while those of $\{\Lambda\}$ have to be determined. $[L]$ can be expressed as known functions of a set of primary variables (b_l). The forcing vector $\{q\}$ is random in nature and can be expressed as known functions of set of loads (q_k). Elements of $\{\Lambda\}$ are unknown and also are the functions of (b_l) and (q_k). The problem is therefore, is to find the statistics of $\{\Lambda\}$ when the statistics of the primary random variables (b_l) and the statistics of the applied loads (q_k) are known.

2.7 RANDOM SYSTEM PROPERTIES AND LOADING

The randomness in material properties and loading are taken into account using 'perturbation technique'. It is assumed that all the primary variables and loading

components are independent of each other. Also it is assumed that the dispersion (randomness) of each random variable about its mean is small in comparison to the mean value. This is common to most engineering problems.

Any random variable can be expressed as a zero mean random part superimposed over a mean. This can be expressed as

$$RP = RP^d + RP^r \tag{2.26}$$

The random process equation $[L]\{\Lambda\} = \{q\}$ can be represented in above form. The basic random process as well as the loading can also be expressed in this manner.

Thus we have

$$[L] = [L^d] + [L^r]$$

$$\{q\} = \{q^d\} + \{q^r\}$$

$$\{\Lambda\} = \{\Lambda^d\} + \{\Lambda^r$$

$$b_l = b_l^d + b_l^r \tag{2.27}$$

where b_l = basic random variable.

As mentioned earlier, for many practical engineering situations the random component (RP^r) is small compared to mean (RP^d). Therefore the equation (2.27) can be written as follows

$$[L_{ij}] = [L_{ij}^d] + \varepsilon [L_{ij}^r]$$

$$\{q_i\} = \{q_i^d\} + \varepsilon \{q_i^r$$

$$\{\Lambda_j\} = \{\Lambda_j^d\} + \varepsilon \{\Lambda_j^r\}$$

$$b_l = b_l^d + \varepsilon b_l^r \tag{2.28}$$

Here 'ε' is a perturbation, small in magnitude. No generality is lost in using the same value of 'ε' in all the cases. The remaining factor in the random term can absorb varying levels of this term.

Using Taylor's series expansion, some of the terms of equation (2.28) can be expanded as

$$\varepsilon L_{ij}{}^{r} = \sum_{l} \frac{\partial L_{ij}{}^{d}}{\partial b_{l}{}^{d}} \varepsilon b_{l}{}^{r}$$

$$\varepsilon \Lambda_{j}{}^{r} = \sum_{l} \frac{\partial \Lambda_{j}{}^{d}}{\partial bl^{d}} \varepsilon b_{l}{}^{r} + \sum_{k} \frac{\partial \Lambda_{j}{}^{d}}{\partial q_{k}{}^{d}} \varepsilon q_{k}{}^{r} \tag{2.29}$$

As $\varepsilon b_{l}{}^{r}$ are small in magnitude, the second and higher order terms are neglected here. This assumption is justified, if the randomness is small compared to mean value and sets a limit to the range of applicability of the approach.

Substituting equation (2.28) in equation (2.25)

$$\left[L^{d} + \varepsilon L^{r}\right]\left\{\Lambda^{d} + \varepsilon \Lambda^{r}\right\} = \left\{q^{d} + \varepsilon q^{r}\right\} \tag{2.30}$$

Expanding equation (2.30) and comparing terms of each order of 'ε', we get

$$\left[L^{d}\Lambda^{d} + \varepsilon L^{d}\Lambda^{r} + \varepsilon L^{r}\Lambda^{d} + \varepsilon^{2} L^{r}\Lambda^{r}\right] = \left\{q^{d} + \varepsilon q^{r}\right\}$$

Collecting terms of different orders of ε, we get for zero$^{\text{th}}$ order and first order terms,

$$\varepsilon^{0} \longrightarrow \qquad [L]^{d}\{\Lambda\}^{d} = \{q\}^{d}$$

$$\varepsilon^{1} \longrightarrow \qquad [L]^{d}\{\Lambda\}^{r} + [L]^{r}\{\Lambda\}^{d} = \{q\}^{r} \tag{2.31}$$

Substituting equation (2.29) in equation (2.31)

$$\sum_{j} L_{ij}{}^{d}\left[\sum_{l} \frac{\partial \Lambda_{j}{}^{d}}{\partial bl} bl^{r} + \sum_{k} \frac{\partial \Lambda_{j}{}^{d}}{\partial q_{k}} q_{k}{}^{r}\right] + \sum_{j}\left[\sum_{l} \frac{\partial L_{ij}{}^{d}}{\partial b_{l}} b_{l}{}^{r}\right] A_{j}{}^{d} = q_{i}{}^{r} \tag{2.32}$$

Since dispersion about mean is small, the derivatives of the random variables can be approximated by the derivatives of the mean values. Then equation (2.32) becomes

$$\sum_j L_{ij}{}^d\left[\sum_l \frac{\partial \Lambda_j{}^d}{\partial b_l{}^d} b_l{}^r + \sum_k \frac{\partial \Lambda_j{}^d}{\partial q_k{}^d} q_k{}^r\right] + \sum_j\left[\sum_l \frac{\partial L_{ij}{}^d}{\partial b_l{}^d} b_l{}^r\right]\Lambda_j{}^d \qquad = q_i{}^r \tag{2.33}$$

Here the unknowns are $\frac{\partial \Lambda_j{}^d}{\partial b_l{}^d}$ and $\frac{\partial \Lambda_j{}^d}{\partial q_k{}^d}$.

To find $\frac{\partial \Lambda_j{}^d}{\partial b_l{}^d}$, multiplying equation (2.33) by $b_u{}^r$ and taking expectation, we get

$$\sum_j\sum_l L_{ij}{}^d \frac{\partial \Lambda_j{}^d}{\partial b_l{}^d} K_{b_l b_u} + \sum_j\sum_k L_{ij}{}^d \frac{\partial \Lambda_j{}^d}{\partial q_k{}^d} K_{q_k b_u} + \sum_j\sum_l A_j{}^d \frac{\partial L_{ij}{}^d}{\partial b_l{}^d} K_{b_l b_u} = K_{q_i b_u} \tag{2.34}$$

K_{xy} is the cross correlation between x and y.

Assuming that the primary variables and the loading components are independent random variables, the above equation becomes

$$\sum_j L_{ij}{}^d \frac{\partial \Lambda_j{}^d}{\partial b_u{}^d} K_{b_u b_u} + \sum_j \Lambda_j{}^d \frac{\partial L_{ij}{}^d}{\partial b_u{}^d} K_{b_u b_u} = 0 \tag{2.35}$$

This is satisfied only when

$$\sum_j L_{ij}{}^d \frac{\partial \Lambda_j{}^d}{\partial b_u{}^d} + \sum_j \Lambda_j{}^d \frac{\partial L_{ij}{}^d}{\partial b_u{}^d} = 0 \tag{2.36}$$

Solving the above equation, we will get $\frac{\partial \Lambda_j{}^d}{\partial b_l{}^d}$.

To find $\frac{\partial \Lambda_j{}^d}{\partial q_k{}^d}$, multiplying equation (2.33) by $q_u{}^r$ and taking expectation, we get

$$\sum_j\sum_k L_{ij}{}^d \frac{\partial \Lambda_j{}^d}{\partial q_k{}^d} K_{q_k q_k} = K_{q_i q_k} \tag{2.37}$$

Again with assumption that $q_u's$ and $q_k's$ are independent of each other, the above equation becomes $\sum_j L_{ij}{}^d \frac{\partial \Lambda_j{}^d}{\partial q_u{}^d} K_{q_k q_k} = K_{q_i q_k}$ (2.38)

Solving the above equation, we get $\dfrac{\partial \Lambda_j^{\,d}}{\partial q_u^{\,d}}$.

Therefore, the total response becomes

$$\Lambda_j = \Lambda_j^{\,d} + \varepsilon\left[\sum_l b_l^{\,r}\,\frac{\partial \Lambda_j^{\,d}}{\partial b_l^{\,d}} + \sum_k q_k^{\,r}\,\frac{\partial \Lambda_j^{\,d}}{\partial q_k^{\,d}}\right] \tag{2.39}$$

The variance of the response can be evaluated as follows

$$\mathrm{Var}\,(\Lambda_j) = E\left[\left(\Lambda_j - \Lambda_j^{\,d}\right)^2\right]$$

$$\mathrm{Var}\,(\Lambda_j) = E\left[\left(\sum_l b_l^{\,r}\,\frac{\partial \Lambda_j^{\,d}}{\partial b_l^{\,d}} + \sum_k q_k^{\,r}\,\frac{\partial \Lambda_j^{\,d}}{\partial q_k^{\,d}}\right)^2\right] \tag{2.40}$$

Standard deviation is the square root of the variance. The above approach is general in nature, and can be applicable to different problems in analysis of structures.

CHAPTER 3

STABILITY AND DYNAMIC ANALYSIS

3.1 INTRODUCTION

This chapter presents the general approach for stability, free and forced vibration analyses of laminated sandwich composite shells with random material properties and external loading.

3.2 BUCKLING OF SHELLS

The increasing use of high strength and stiffness materials in recent times for structural component design and the need to produce light weight and optimized structures have lead to widespread adoption of thin walled components in aerospace structures. The aerospace structures, generally made up of lightweight materials, experience complex loading during service life. Instability may occur in these structures, leads to buckling failure. To avoid this, evaluation of critical buckling load is an essential requirement for design of optimized structural components. A unified approach is discussed for buckling analyses of cylinder, sphere and conical shells that involve anisotropic, layered composite construction.

This section outlines a probabilistic methodology for application of classical approach in conjunction with first order perturbation technique (FOPT) for evaluation of

second order statistics of critical buckling load (mean and SD) as a function of the basic random variables - Longitudinal modulus, transverse modulus and Poisson's ratio.

Classical thin plate theory has been adopted for the analysis. According to this theory, a linear element of the shell, extending through the wall thickness and normal to the middle surface in the unstrained state, upon the application of load satisfies the following conditions:

- Undergoes at most a translation and a rotation with respect to the original coordinate system.
- Remains normal to the deformed middle surface.
- It is assumed that shell resists lateral and in-plane loads by bending, transverse shear by membrane action and not through block compression or tension of the shell wall in the thickness direction.
- The linear element does not elongate nor contracts.
- It remains straight upon the shell being loaded.
- Saint Venant principle applies.

The main assumptions in the classical laminate theory may be put as

- Thickness of the panel is small compared to its other lateral dimensions.
- Plate deflections are small compared to its thickness.
- Normal stresses in the transverse direction can be neglected.

For buckling of the shell under external loading, it is assumed that pre buckled deformation are not taken into account and ends of the cylindrical shell are supported by rings rigid in their planes but with no resistance to rotation or bending out of their plane.

3.2.1 STOCHASTIC CLASSICAL APPROACH

For all edges are simply supported, exact Navier type solution is possible for curved panels. When exact solution is combined with probabilistic method, it is possible to analyze the random characteristics equation arising from buckling of composite laminates. In this section, a detailed study is presented to obtain the second order statistics of the buckling loads for sandwich cylindrical, spherical, and conical panels using stochastic classical approach (SCA).

Boundary conditions are:

Referring to the figure1.1a

$$
\begin{aligned}
x=0,a \quad & v(0,y) = v(a,y) = 0 \\
& w(0,y) = w(a,y) = 0 \\
& N_1(0,y) = N_2(a,y) = 0 \\
y=0,b \quad & u(x,0) = u(x,b) = 0 \\
& w(x,0) = w(x,b) = 0 \\
& N_1(x,0) = N_2(x,0) = 0
\end{aligned} \tag{3.1}
$$

The displacements satisfying all the boundary conditions can be expressed as

$$
\begin{aligned}
u(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \\
v(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \\
w(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}
\end{aligned} \tag{3.2}
$$

Substituting the equation (3.2) into equation (2.25) results in a homogeneous system of equations

$$\sum_{j=i}^{n} M_{ij} k_j = 0 \tag{3.3}$$

where 'k' is a constant column vector, i, j=1,2 and 3 for CLT. For nontrivial solution of 'k' from equation (3.3), the determinant of coefficients should be zero:

$$|M_{ij}| = 0 \tag{3.4}$$

Equation (3.4) can be written in expanded form as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} - N_{cr} \end{vmatrix} = 0 \tag{3.5}$$

Where the expression for critical buckling load is given as

$$N_{cr} = N_x(m^2\pi^2 / a^2) + N_{y'}(n^2\pi^2 / b^2) + N_{xy'}(mn\pi^2 / ab) \tag{3.6}$$

In the above equation, a_{ij}'s are functions of system stiffness, constants "m" and "n".

Expanding the above determinant gives the expression for N_{cr} in terms of a_{ij}'s.

$$N_{cr}=F\ (a_{ij}) \tag{3.7}$$

The stiffness elements a_{ij} are random in nature, being dependent on the system material properties, consequently, the buckling loads are also random.

3.2.2 SECOND ORDER STATISTICS OF CRITICAL BUCKLING LOADS – PERTURBATION APPROACH

A random variable can be split up as the sum of its mean and the zero mean random part

$$N_{cr} = N_{cr}^{\ d} + N_{cr}^{\ r} \tag{3.8}$$

Similarly, a_{ij} can also be expressed as $a_{ij} = a_{ij}^{\ d} + a_{ij}^{\ r}$ (3.9)

On substitution of (3.8) and (3.9) in (3.5), yields

$$\begin{vmatrix} a_{11}{}^{d} + a_{11}{}^{r} & a_{12}{}^{d} + a_{12}{}^{r} & a_{13}{}^{d} + a_{13}{}^{r} \\ a_{21}{}^{d} + a_{21}{}^{r} & a_{22}{}^{d} + a_{22}{}^{r} & a_{23}{}^{d} + a_{13}{}^{r} \\ a_{31}{}^{d} + a_{31}{}^{r} & a_{32}{}^{d} + a_{32}{}^{d} & a_{33}{}^{d} + a_{33}{}^{r} - N_{cr}{}^{d} - N_{cr}{}^{r} \end{vmatrix} = 0 \tag{3.10}$$

Expanding (3.10), collecting the same order of magnitude terms, and keeping only up to first-order terms, one obtains the following relation in the symbolic form:

Zeroth order:

$$N_{cr}{}^{d} = F\,(a_{ij}{}^{d}) \tag{3.11}$$

First order:

$$N_{cr}{}^{r} = F\,(a_{ij}{}^{d},\ a_{ij}{}^{r},\ N_{cr}{}^{d}) \tag{3.12}$$

Equation (3.11) is a deterministic equation relating the mean quantities. The mean value of buckling loads can be obtained by solving the following expression

$$N_{cr}{}^{d} = \frac{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} - N_{cr} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \tag{3.13}$$

Equation (3.11) gives an expansion

$$\begin{aligned} N_{cr}{}^{r} = ((& a_{11}{}^{d} a_{33}{}^{d} a_{22}{}^{r} + a_{22}{}^{d} a_{33}{}^{d} a_{11}{}^{r} + a_{11}{}^{d} a_{22}{}^{d} a_{33}{}^{r} + a_{12}{}^{d} a_{23}{}^{d} a_{13}{}^{r} + 2a_{13}{}^{d} a_{23}{}^{d} a_{12}{}^{r} + 2a_{12}{}^{d} a_{13}{}^{d} a_{23}{}^{r} \\ & - 2a_{11}{}^{d} a_{23}{}^{d} a_{23}{}^{r} - a_{11}{}^{r} a_{23}{}^{d2} + 2a_{12}{}^{d} a_{23}{}^{d} a_{13}{}^{r} - a_{22}{}^{r} a_{13}{}^{d2} - 2a_{22}{}^{d} a_{13}{}^{d} a_{13}{}^{r} - a_{33}{}^{r} a_{12}{}^{d2}) \\ & - N_{cr}{}^{d} (a_{11}{}^{d} a_{22}{}^{r} + a_{11}{}^{r} a_{22}{}^{d} - 2a_{12}{}^{d} a_{12}{}^{r})) / G \end{aligned}$$

Where $G = (a_{11}{}^{d} a_{22}{}^{d} - a_{12}{}^{d2})$ (3.14)

Equation (3.12) is the random equation, which is a function of $a_{ij}{}^{d}$, $a_{ij}{}^{r}$ and $N_{cr}{}^{d}$. For present case, $a_{ij}{}^{r}$ and $N_{cr}{}^{r}$ are random because the material properties are random, as

discussed earlier. Let $b_1^r, b_2^r \ldots b_m^r$ denote the random material properties. The b_l^r can also be expressed as

$$b_l = b_l^d + b_l^r \qquad (3.15)$$

Expanding the above expression in Taylor's series, when b_l^r is small compared with its mean value, one can expand L_{ij}^r and N_{cr}^r about b_l^d=1,2...m. keeping only the first-order terms, one obtains

$$N_{cr}^r = \sum_{l=1}^{m} N_{cr,l}{}^d b_l^r \qquad a_{ij}^r = \sum_{l=1}^{m} a_{ik,l}{}^d b_l^r \qquad (3.16)$$

Where 'j' denotes the partial differentiation with respect to b_l ; and the derivatives are evaluated at b_l^d.

$b_1=E_{11}$, $b2=E_{22}$ and $b_3=\upsilon_{12}$

Substituting (3.16), in to (3.14) and simplifying, we get

$$N_{cr}^r = PE_{11}^r + QE_{22}^r + R\upsilon_{12}^r \qquad (3.17)$$

From basic definition of variance,

$$Var(N_{cr}) = E\left[\sum_{l=1}^{m} N_{cr}{}^d{}_{,l} b_l^r \sum_{k=1}^{m} N_{cr}{}^d{}_{,k} b_k^r\right] = \sum_{l=1}^{m}\sum_{k=1}^{m} N_{cr}{}^d{}_{,l}\, N_{cr}{}^d{}_{,k} E\left[b_l^r b_k^r\right]$$

$$= \sum_{l=1}^{m}\sum_{k=1}^{m} N_{cr}{}^d{}_{,l}\, N_{cr}{}^d{}_{,k} \operatorname{cov}(b_l^r, b_k^r) \qquad (3.18)$$

This results

$$\operatorname{var}(N_{cr}^r) = p^2\left[\sigma_{E_{11}^r}\right] + Q^2\left[\sigma_{E_{22}^r}\right] + R^2\left[\sigma_{\upsilon_{12}^r}\right. \qquad (3.19)$$

SD (N_{cr}^r)=Sqrt (var (N_{cr}^r)) (3.20)

where

$$P = \frac{1}{G}[\{a_{11}{}^{d} a_{33}{}^{d} \frac{\partial a_{22}{}^{d}}{\partial E_{11}{}^{d}} + a_{22}{}^{d} a_{33}{}^{d} \frac{\partial a_{11}{}^{d}}{\partial E_{11}{}^{d}} + a_{11}{}^{d} a_{22}{}^{d} \frac{\partial a_{33}{}^{d}}{\partial E_{11}{}^{d}} + 2a_{12}{}^{d} a_{23}{}^{d} \frac{\partial a_{13}{}^{d}}{\partial E_{11}{}^{d}}$$

$$+ 2a_{12}{}^{d} a_{13}{}^{d} \frac{\partial a_{23}{}^{d}}{\partial E_{11}{}^{d}} + 2a_{23}{}^{d} a_{13}{}^{d} \frac{\partial a_{12}{}^{d}}{\partial E_{11}{}^{d}} - 2a_{12}{}^{d} a_{33}{}^{d} \frac{\partial a_{12}{}^{d}}{\partial E_{11}{}^{d}} - 2a_{23}{}^{d} a_{11}{}^{d} \frac{\partial a_{23}{}^{d}}{\partial E_{11}{}^{d}}$$

$$- 2a_{13}{}^{d} a_{22}{}^{d} \frac{\partial a_{13}{}^{d}}{\partial E_{11}{}^{d}} - a_{23}{}^{d2} \frac{\partial a_{11}{}^{d}}{\partial E_{11}{}^{d}} - a_{13}{}^{d2} \frac{\partial a_{22}{}^{d}}{\partial E_{11}{}^{d}} - a_{12}{}^{d2} \frac{\partial a_{33}{}^{d}}{\partial E_{11}{}^{d}}\}$$

$$- N_{cr}{}^{d} \{a_{11}{}^{d} \frac{\partial a_{22}{}^{d}}{\partial E_{11}{}^{d}} + a_{22}{}^{d} \frac{\partial a_{11}{}^{d}}{\partial E_{11}{}^{d}} - 2a_{12}{}^{d} \frac{\partial a_{12}{}^{d}}{\partial E_{11}{}^{d}}\}]$$

$$Q = \frac{1}{G}[\{a_{11}{}^{d} a_{33}{}^{d} \frac{\partial a_{22}{}^{d}}{\partial E_{22}{}^{d}} + a_{22}{}^{d} a_{33}{}^{d} \frac{\partial a_{11}{}^{d}}{\partial E_{22}{}^{d}} + a_{11}{}^{d} a_{22}{}^{d} \frac{\partial a_{33}{}^{d}}{\partial E_{22}{}^{d}} + 2a_{12}{}^{d} a_{23}{}^{d} \frac{\partial a_{13}{}^{d}}{\partial E_{22}{}^{d}}$$

$$+ 2a_{12}{}^{d} a_{13}{}^{d} \frac{\partial a_{23}{}^{d}}{\partial E_{22}{}^{d}} + 2a_{23}{}^{d} a_{13}{}^{d} \frac{\partial a_{12}{}^{d}}{\partial E_{22}{}^{d}} - 2a_{12}{}^{d} a_{33}{}^{d} \frac{\partial a_{12}{}^{d}}{\partial E_{22}{}^{d}} - 2a_{23}{}^{d} a_{11}{}^{d} \frac{\partial a_{23}{}^{d}}{\partial E_{22}{}^{d}}$$

$$- 2a_{13}{}^{d} a_{22}{}^{d} \frac{\partial a_{13}{}^{d}}{\partial E_{22}{}^{d}} - a_{23}{}^{d2} \frac{\partial a_{11}{}^{d}}{\partial E_{22}{}^{d}} - a_{13}{}^{d2} \frac{\partial a_{22}{}^{d}}{\partial E_{22}{}^{d}} - a_{12}{}^{d2} \frac{\partial a_{33}{}^{d}}{\partial E_{22}{}^{d}}\}$$

$$- N_{cr}{}^{d} \{a_{11}{}^{d} \frac{\partial a_{22}{}^{d}}{\partial E_{22}{}^{d}} + a_{22}{}^{d} \frac{\partial a_{11}{}^{d}}{\partial E_{22}{}^{d}} - 2a_{12}{}^{d} \frac{\partial a_{12}{}^{d}}{\partial E_{22}{}^{d}}\}]$$

$$R = \frac{1}{G}[\{a_{11}{}^{d} a_{33}{}^{d} \frac{\partial a_{22}{}^{d}}{\partial \upsilon_{12}{}^{d}} + a_{22}{}^{d} a_{33}{}^{d} \frac{\partial a_{11}{}^{d}}{\partial \upsilon_{12}{}^{d}} + a_{11}{}^{d} a_{22}{}^{d} \frac{\partial a_{33}{}^{d}}{\partial \upsilon_{12}{}^{d}} + 2a_{12}{}^{d} a_{23}{}^{d} \frac{\partial a_{13}{}^{d}}{\partial \upsilon_{12}{}^{d}}$$

$$+ 2a_{12}{}^{d} a_{13}{}^{d} \frac{\partial a_{23}{}^{d}}{\partial \upsilon_{12}{}^{d}} + 2a_{23}{}^{d} a_{13}{}^{d} \frac{\partial a_{12}{}^{d}}{\partial \upsilon_{12}{}^{d}} - 2a_{12}{}^{d} a_{33}{}^{d} \frac{\partial a_{12}{}^{d}}{\partial \upsilon_{12}{}^{d}} - 2a_{23}{}^{d} a_{11}{}^{d} \frac{\partial a_{23}{}^{d}}{\partial \upsilon_{12}{}^{d}}$$

$$- 2a_{13}{}^{d} a_{22}{}^{d} \frac{\partial a_{13}{}^{d}}{\partial \upsilon_{12}{}^{d}} - a_{23}{}^{d2} \frac{\partial a_{11}{}^{d}}{\partial \upsilon_{12}{}^{d}} - a_{13}{}^{d2} \frac{\partial a_{22}{}^{d}}{\partial \upsilon_{12}{}^{d}} - a_{12}{}^{d2} \frac{\partial a_{33}{}^{d}}{\partial \upsilon_{12}{}^{d}}\}$$

$$- N_{cr}{}^{d} \{a_{11}{}^{d} \frac{\partial a_{22}{}^{d}}{\partial \upsilon_{12}{}^{d}} + a_{22}{}^{d} \frac{\partial a_{11}{}^{d}}{\partial \upsilon_{12}{}^{d}} - 2a_{12}{}^{d} \frac{\partial a_{12}{}^{d}}{\partial \upsilon_{12}{}^{d}}\}] \tag{3.21}$$

3.3 FREEVIBRATIONS:

The free vibration equation is obtained by setting the in-plane loads and excitation terms to zero in the governing equation (2.10).

For all edges are simply supported, exact Navier type solution is possible. When exact solution is combined with probabilistic method, it is possible to analyze the random generalized eigen value problem associated with free vibration of composite laminated panels with random material properties.

In this section, a detailed study is presented to obtain the second order statistics of the natural frequencies for sandwich honeycombed cylindrical, spherical, and conical panels using stochastic classical approach.

(3.22)

The displacements satisfying all the boundary conditions can be expressed as

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\exp(i\omega t)$$
$$v(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\exp(i\omega t)$$
$$w(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\exp(i\omega t) \qquad (3.23)$$

Substituting the equation (3.23) into equation (2.25) results in a homogeneous system of equations

$$\sum_{j=i}^{n}M_{ij}k_j=0 \qquad (3.24)$$

Where **k** is a constant column vector, i, j=1,2 and 3 for CLT. For nontrivial solution of '**k**' ,In equation (3.24), the determinant of coefficients should be zero:

$$\left|M_{ij}\right|=0 \qquad (3.25)$$

Equation (3.25) can be written in expanded form as

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}=0 \qquad (3.26)$$

The above equation can be expanded as

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{3.27}$$

where $\lambda = \rho\omega^2$

$$I_1 = a_{11} + a_{22} + a_{33}$$

$$I_2 = a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{12}a_{21} - a_{23}a_{23} - a_{13}a_{31}$$

$$I_3 = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} \tag{3.28}$$

To obtain the statistics of the natural frequency, the variable terms can be written as sum of the mean terms and the corresponding zero mean random terms.

$$\lambda = \lambda^d + \lambda^r$$

Likewise, all other random quantities are defined as follows

$$a_{11} = a_{11}{}^d + a_{11}{}^r \qquad a_{22} = a_{22}{}^d + a_{22}{}^r \qquad a_{33} = a_{33}{}^d + a_{33}{}^r$$

$$a_{12} = a_{12}{}^d + a_{12}{}^r \qquad a_{13} = a_{13}{}^d + a_{13}{}^r \qquad a_{23} = a_{23}{}^d + a_{23}{}^r$$

Substituting these in the above equation (3.27)

$$\left(\lambda^{d^3} + \lambda^{r^3} + 3\lambda^d\lambda^{r^2} + 3\lambda^{d^2}\lambda^r\right) - \left(\lambda^{d^2} + \lambda^{r^2} + 2\lambda^d\lambda^r\right)\left[a_{11}{}^d + a_{11}{}^r + a_{22}{}^d + a_{22}{}^r + a_{33}{}^d + a_{33}{}^r\right]$$

$$+\left(\lambda^d + \lambda^r\right)\{a_{11}{}^d a_{22}{}^d + a_{11}{}^d a_{22}{}^r + a_{11}{}^r a_{22}{}^d + a_{11}{}^r a_{22}{}^r + a_{11}{}^d a_{33}{}^d + a_{11}{}^d a_{33}{}^r + a_{11}{}^r a_{33}{}^d + a_{11}{}^r a_{33}{}^r$$

$$+ a_{22}{}^d a_{33}{}^d + a_{22}{}^d a_{33}{}^r + a_{22}{}^r a_{33}{}^d + a_{22}{}^r a_{33}{}^r - a_{12}{}^d a_{21}{}^d - a_{12}{}^d a_{21}{}^r - a_{12}{}^r a_{21}{}^d - a_{12}{}^r a_{21}{}^r$$

$$- a_{23}{}^d a_{32}{}^d - a_{23}{}^d a_{32}{}^r - a_{23}{}^r a_{32}{}^d - a_{23}{}^r a_{32}{}^r - a_{13}{}^d a_{31}{}^d - a_{13}{}^d a_{31}{}^r - a_{13}{}^r a_{31}{}^d - a_{13}{}^r a_{31}{}^r\}$$

$$-\{(a_{11}{}^{d}a_{22}{}^{d}a_{33}{}^{d}+a_{11}{}^{d}a_{22}{}^{r}a_{33}{}^{d}+a_{11}{}^{r}a_{22}{}^{d}a_{33}{}^{d}+a_{11}{}^{r}a_{22}{}^{r}a_{33}{}^{d}+a_{11}{}^{d}a_{22}{}^{d}a_{33}{}^{r}+a_{11}{}^{d}a_{22}{}^{r}a_{33}{}^{r}$$

$$+a_{11}{}^{r}a_{22}{}^{d}a_{33}{}^{r}+a_{11}{}^{r}a_{22}{}^{r}a_{33}{}^{r})-(a_{11}{}^{d}a_{23}{}^{d}a_{32}{}^{d}+a_{11}{}^{d}a_{23}{}^{r}a_{32}{}^{d}+a_{11}{}^{r}a_{23}{}^{d}a_{32}{}^{d}+a_{11}{}^{r}a_{23}{}^{r}a_{32}{}^{d}$$

$$+a_{11}{}^{d}a_{23}{}^{d}a_{32}{}^{r}+a_{11}{}^{d}a_{23}{}^{r}a_{32}{}^{r}+a_{11}{}^{r}a_{23}{}^{d}a_{32}{}^{r}+a_{11}{}^{r}a_{23}{}^{r}a_{32}{}^{r})-(a_{12}{}^{d}a_{21}{}^{d}a_{33}{}^{d}+a_{12}{}^{d}a_{21}{}^{r}a_{33}{}^{d}$$

$$+a_{12}{}^{r}a_{21}{}^{d}a_{33}{}^{d}+a_{12}{}^{r}a_{21}{}^{r}a_{33}{}^{d}+a_{12}{}^{d}a_{21}{}^{d}a_{33}{}^{r}+a_{12}{}^{d}a_{21}{}^{r}a_{33}{}^{r}+a_{12}{}^{r}a_{21}{}^{d}a_{33}{}^{r}+a_{12}{}^{r}a_{21}{}^{r}a_{33}{}^{r})+$$

$$(a_{12}{}^{d}a_{23}{}^{d}a_{31}{}^{d}+a_{12}{}^{d}a_{23}{}^{r}a_{31}{}^{d}+a_{12}{}^{r}a_{23}{}^{d}a_{31}{}^{d}+a_{12}{}^{r}a_{23}{}^{r}a_{31}{}^{d}+a_{12}{}^{d}a_{23}{}^{d}a_{31}{}^{r}+a_{12}{}^{d}a_{23}{}^{r}a_{31}{}^{r})\}$$

(3.29)

Taking expectation of above equation, we get

$$\lambda^{d^3}-\lambda^{d^2}\{a_{11}{}^{d}+a_{22}{}^{d}+a_{33}{}^{d}\}+\lambda^{d}\{a_{11}{}^{d}a_{22}{}^{d}+a_{11}{}^{d}a_{33}{}^{d}+a_{22}{}^{d}a_{33}{}^{d}+-a_{12}{}^{d}a_{21}{}^{d}-a_{23}{}^{d}a_{32}{}^{d}-a_{13}{}^{d}a_{31}{}^{d}\}$$

$$-\{a_{11}{}^{d}a_{22}{}^{d}a_{33}{}^{d}-a_{11}{}^{d}a_{23}{}^{d}a_{32}{}^{d}-a_{12}{}^{d}a_{21}{}^{d}a_{33}{}^{d}+a_{12}{}^{d}a_{23}{}^{d}a_{31}{}^{d}+a_{13}{}^{d}a_{21}{}^{d}a_{32}{}^{d}-a_{31}{}^{d}a_{22}{}^{d}a_{13}{}^{d}\}=0$$

(3.30)

As, the expectation of all random terms is zero. The above equation is the characteristic equation for the mean part of the random natural frequencies. Subtracting the equation (3.30) from (3.29), and on simplifying, we get

$$\lambda^{r}=\frac{1}{G}\{\lambda^{d^2}(a_{11}{}^{r}+a_{22}{}^{r}+a_{33}{}^{r})-\lambda^{d}(a_{11}{}^{d}a_{22}{}^{r}+a_{11}{}^{r}a_{22}{}^{d}+a_{11}{}^{d}a_{33}{}^{r}+a_{11}{}^{r}a_{33}{}^{d}+a_{22}{}^{d}a_{33}{}^{r}$$

$$+a_{22}{}^{r}a_{33}{}^{d}-a_{12}{}^{d}a_{21}{}^{r}-a_{12}{}^{r}a_{21}{}^{d}-a_{23}{}^{d}a_{32}{}^{r}-a_{23}{}^{r}a_{32}{}^{d}-a_{13}{}^{d}a_{31}{}^{r}-a_{13}{}^{r}a_{31}{}^{d})+F\}$$

(3.31)

where

$$G=3\lambda^{d^2}-2\lambda^{d}(a_{11}{}^{d}+a_{22}{}^{d}+a_{33}{}^{d})$$

$$+(a_{11}{}^{d}a_{22}{}^{d}+a_{11}{}^{d}a_{33}{}^{d}+a_{22}{}^{d}a_{33}{}^{d}-a_{12}{}^{d}a_{21}{}^{d}-a_{23}{}^{d}a_{32}{}^{d}-a_{13}{}^{d}a_{31}{}^{d})$$

$$F = a_{11}{}^{d} a_{33}{}^{d} a_{22}{}^{r} + a_{22}{}^{d} a_{33}{}^{d} a_{11}{}^{r} + a_{11}{}^{d} a_{22}{}^{d} a_{33}{}^{r} - a_{11}{}^{d} a_{32}{}^{d} a_{23}{}^{r} - a_{23}{}^{d} a_{32}{}^{d} a_{11}{}^{r} - a_{11}{}^{d} a_{23}{}^{d} a_{32}{}^{r}$$

$$- a_{12}{}^{d} a_{33}{}^{d} a_{21}{}^{r} - a_{21}{}^{d} a_{33}{}^{d} a_{12}{}^{r} - a_{12}{}^{d} a_{21}{}^{d} a_{33}{}^{r} + a_{12}{}^{d} a_{31}{}^{d} a_{23}{}^{r} + a_{23}{}^{d} a_{31}{}^{d} a_{12}{}^{r} + a_{12}{}^{d} a_{23}{}^{d} a_{31}{}^{r}$$

$$- a_{13}{}^{d} a_{32}{}^{d} a_{21}{}^{r} + a_{21}{}^{d} a_{32}{}^{d} a_{13}{}^{r} + a_{13}{}^{d} a_{21}{}^{d} a_{32}{}^{r} - a_{31}{}^{d} a_{13}{}^{d} a_{22}{}^{r} - a_{22}{}^{d} a_{13}{}^{d} a_{31}{}^{r} - a_{31}{}^{d} a_{22}{}^{d} a_{13}{}^{r}$$

(3.32)

$a_{ij}{}^{r}$ can be expressed as a function of input random variables, through Taylor's series expansion as

$$a_{ij}{}^{r} = \sum_{l} \frac{\partial a_{ij}{}^{d}}{\partial b_l{}^{d}} b_l{}^{r} = \sum_{l=1}^{3} \frac{\partial a_{ij}{}^{d}}{\partial bl^{d}} b_l{}^{r} \tag{3.33}$$

Taking $b_1 = E_{11}, b_2 = E_{22}$ and $b_3 = \upsilon_{12}$

Using above equation, and simplifying, we have

$$\lambda^{r} = P\left[E_{11}{}^{r}\right] + Q\left[E_{22}{}^{r}\right] + R\left[\upsilon_{12}{}^{r}\right] \tag{3.34}$$

where

$$P = \frac{1}{G}[\lambda^{d^2}(\frac{\partial a_{11}{}^{d}}{\partial E_{11}{}^{d}} + \frac{\partial a_{22}{}^{d}}{\partial E_{11}{}^{d}} + \frac{\partial a_{33}{}^{d}}{\partial E_{11}{}^{d}}) - \lambda^{d}\{(a_{11}{}^{d}\frac{\partial a_{22}{}^{d}}{\partial E_{11}{}^{d}} + \frac{\partial a_{33}{}^{d}}{\partial E_{11}{}^{d}}) + a_{22}{}^{d}(\frac{\partial a_{11}{}^{d}}{\partial E_{11}{}^{d}} + \frac{\partial a_{33}{}^{d}}{\partial E_{11}{}^{d}})$$
$$+ a_{33}{}^{d}(\frac{\partial a_{11}{}^{d}}{\partial E_{11}{}^{d}} + \frac{\partial a_{22}{}^{d}}{\partial E_{11}{}^{d}}) - 2(a_{12}{}^{d}\frac{\partial a_{12}{}^{d}}{\partial E_{11}{}^{d}} + a_{23}{}^{d}\frac{\partial a_{23}{}^{d}}{\partial E_{11}{}^{d}} + a_{13}{}^{d}\frac{\partial a_{13}{}^{d}}{\partial E_{11}{}^{d}})\} + \{a_{11}{}^{d}a_{33}{}^{d}\frac{\partial a_{22}{}^{d}}{\partial E_{11}{}^{d}}$$
$$+ a_{22}{}^{d}a_{33}{}^{d}\frac{\partial a_{11}{}^{d}}{\partial E_{11}{}^{d}} + a_{11}{}^{d}a_{22}{}^{d}\frac{\partial a_{33}{}^{d}}{\partial E_{11}{}^{d}} + 2a_{12}{}^{d}a_{23}{}^{d}\frac{\partial a_{13}{}^{d}}{\partial E_{11}{}^{d}} + 2a_{12}{}^{d}a_{13}{}^{d}\frac{\partial a_{23}{}^{d}}{\partial E_{11}{}^{d}} + 2a_{23}{}^{d}a_{13}{}^{d}\frac{\partial a_{12}{}^{d}}{\partial E_{11}{}^{d}}$$
$$- 2a_{21}{}^{d}a_{33}{}^{d}\frac{\partial a_{12}{}^{d}}{\partial E_{11}{}^{d}} - 2a_{23}{}^{d}a_{11}{}^{d}\frac{\partial a_{23}{}^{d}}{\partial E_{11}{}^{d}} - 2a_{13}{}^{d}a_{22}{}^{d}\frac{\partial a_{13}{}^{d}}{\partial E_{11}{}^{d}} - a_{23}{}^{d^2}\frac{\partial a_{11}{}^{d}}{\partial E_{11}{}^{d}}$$
$$- a_{13}{}^{d^2}\frac{\partial a_{22}{}^{d}}{\partial E_{11}{}^{d}} - a_{12}{}^{d^2}\frac{\partial a_{33}{}^{d}}{\partial E_{11}{}^{d}}\}]$$

$$Q=\frac{1}{G}[\lambda^{d2}(\frac{\partial a_{11}{}^d}{\partial E_{22}{}^d}+\frac{\partial a_{22}{}^d}{\partial E_{22}{}^d}+\frac{\partial a_{33}{}^d}{\partial E_{22}{}^d})-\lambda^d\{(a_{11}{}^d\frac{\partial a_{22}{}^d}{\partial E_{22}{}^d}+\frac{\partial a_{33}{}^d}{\partial E_{22}{}^d})+a_{22}{}^d(\frac{\partial a_{11}{}^d}{\partial E_{22}{}^d}+\frac{\partial a_{33}{}^d}{\partial E_{22}{}^d})$$

$$+a_{33}{}^d(\frac{\partial a_{11}{}^d}{\partial E_{22}{}^d}+\frac{\partial a_{22}{}^d}{\partial E_{22}{}^d})-2(a_{12}{}^d\frac{\partial a_{12}{}^d}{\partial E_{22}{}^d}+a_{23}{}^d\frac{\partial a_{23}{}^d}{\partial E_{22}{}^d}+a_{13}{}^d\frac{\partial a_{13}{}^d}{\partial E_{22}{}^d})\}+\{a_{11}{}^d a_{33}{}^d\frac{\partial a_{22}{}^d}{\partial E_{22}{}^d}$$

$$+a_{22}{}^d a_{33}{}^d\frac{\partial a_{11}{}^d}{\partial E_{22}{}^d}+a_{11}{}^d a_{22}{}^d\frac{\partial a_{33}{}^d}{\partial E_{22}{}^d}+2a_{12}{}^d a_{23}{}^d\frac{\partial a_{13}{}^d}{\partial E_{22}{}^d}+2a_{12}{}^d a_{13}{}^d\frac{\partial a_{23}{}^d}{\partial E_{22}{}^d}+2a_{23}{}^d a_{13}{}^d\frac{\partial a_{12}{}^d}{\partial E_{22}{}^d}$$

$$-2a_{12}{}^d a_{33}{}^d\frac{\partial a_{12}{}^d}{\partial E_{22}{}^d}-2a_{23}{}^d a_{11}{}^d\frac{\partial a_{23}{}^d}{\partial E_{22}{}^d}-2a_{13}{}^d a_{22}{}^d\frac{\partial a_{13}{}^d}{\partial E_{22}{}^d}-a_{23}{}^{d2}\frac{\partial a_{11}{}^d}{\partial E_{22}{}^d}$$

$$-a_{13}{}^{d2}\frac{\partial a_{22}{}^d}{\partial E_{22}{}^d}-a_{12}{}^{d2}\frac{\partial a_{33}{}^d}{\partial E_{22}{}^d}\}]$$

$$R=\frac{1}{G}[\lambda^{d2}(\frac{\partial a_{11}{}^d}{\partial \upsilon_{12}{}^d}+\frac{\partial a_{22}{}^d}{\partial \upsilon_{12}{}^d}+\frac{\partial a_{33}{}^d}{\partial \upsilon_{12}{}^d})-\lambda^d\{(a_{11}{}^d\frac{\partial a_{22}{}^d}{\partial \upsilon_{12}{}^d}+\frac{\partial a_{33}{}^d}{\partial \upsilon_{12}{}^d})+a_{22}{}^d(\frac{\partial a_{11}{}^d}{\partial \upsilon_{12}{}^d}+\frac{\partial a_{33}{}^d}{\partial \upsilon_{12}{}^d})$$

$$+a_{33}{}^d(\frac{\partial a_{11}{}^d}{\partial \upsilon_{12}{}^d}+\frac{\partial a_{22}{}^d}{\partial \upsilon_{12}{}^d})-2(a_{12}{}^d\frac{\partial a_{12}{}^d}{\partial \upsilon_{12}{}^d}+a_{23}{}^d\frac{\partial a_{23}{}^d}{\partial \upsilon_{12}{}^d}+a_{13}{}^d\frac{\partial a_{13}{}^d}{\partial \upsilon_{12}{}^d})\}+\{a_{11}{}^d a_{33}{}^d\frac{\partial a_{22}{}^d}{\partial \upsilon_{12}{}^d}$$

$$+a_{22}{}^d a_{33}{}^d\frac{\partial a_{11}{}^d}{\partial \upsilon_{12}{}^d}+a_{11}{}^d a_{22}{}^d\frac{\partial a_{33}{}^d}{\partial \upsilon_{12}{}^d}+2a_{12}{}^d a_{23}{}^d\frac{\partial a_{13}{}^d}{\partial \upsilon_{12}{}^d}+2a_{12}{}^d a_{13}{}^d\frac{\partial a_{23}{}^d}{\partial \upsilon_{12}{}^d}+2a_{23}{}^d a_{13}{}^d\frac{\partial a_{12}{}^d}{\partial \upsilon_{12}{}^d}$$

$$-2a_{12}{}^d a_{33}{}^d\frac{\partial a_{12}{}^d}{\partial \upsilon_{12}{}^d}-2a_{23}{}^d a_{11}{}^d\frac{\partial a_{23}{}^d}{\partial \upsilon_{12}{}^d}-2a_{13}{}^d a_{22}{}^d\frac{\partial a_{13}{}^d}{\partial \upsilon_{12}{}^d}-a_{23}{}^{d2}\frac{\partial a_{11}{}^d}{\partial \upsilon_{12}{}^d}$$

$$-a_{13}{}^{d2}\frac{\partial a_{22}{}^d}{\partial \upsilon_{12}{}^d}-a_{12}{}^{d2}\frac{\partial a_{33}{}^d}{\partial \upsilon_{12}{}^d}\}]$$

Taking the second order moment of equation of (3.34), we have

$$E[\lambda^2]=P^2E\left[E_{11}{}^{r2}\right]+Q^2E\left[E_{22}{}^{r2}\right]+R^2E\left[E_{12}{}^{r2}\right]+PQE\left[E_{11}{}^{r}E_{22}{}^{r}\right]+\ldots\ldots$$

It is assumed that all the four zero mean random variables are independent of each other, so that cross-correlation terms vanish.

Thus, the above equation takes the form:

$$E[\lambda^2]=P^2E\left[E_{11}{}^{r2}\right]+Q^2E\left[E_{22}{}^{r2}\right]+R^2E\left[E_{12}{}^{r2}\right] \tag{3.35}$$

By definition of the variance, we have

$$Var(\lambda^{r^2}) = P^2\sigma E_{11}{}^{r^2} + Q^2\sigma E_{22}{}^{r^2} + R^2\sigma U_{12}{}^{r^2} \quad (3.36)$$

$$SD(\lambda^{r^2}) = \sqrt{P^2\sigma E_{11}{}^{r^2} + Q^2\sigma E_{22}{}^{r^2} + R^2\sigma U_{12}{}^{r^2}} \quad (3.37)$$

Thus, using the above equation $SD(\omega^{r^2})$ is obtained.

3.4 FORCED VIBRATIONS

3.4.1 INTRODUCTION

In general, vibration is undesirable for all aerospace structures. When a structure is subjected to external excitation, it is forced to vibrate at the same frequency as that of the excitation. These vibrations may be undesirable for the structure whose functioning may be disturbed or it may lead to its fatigue failure, if large vibration amplitudes develop.

Most aerospace structures operate in random environment during their service life, i.e. the applied excitation is not deterministic. The following are some of the examples for random excitation.

- The gust excitation caused by atmospheric turbulence
- The excitation due to jet and rocket noise
- The aerodynamic excitation due to boundary layer turbulence and fluctuating wake forces.
- The ground loads induced during taxi, takeoff and landing of an aircraft or during the transportation of a launch vehicle to the launch pad.

These results in need of modeling the system properties and excitation as random.

In the present study, the system properties like longitudinal modulus, transverse modulus and Poison's ratio, as well as the excitation are considered to be random.

In this section, a detailed study is presented to obtain the second order statistics of the response for sandwich honeycombed cylindrical, spherical, and conical panels using stochastic classical approach adopting the first order perturbation theory.

3.4.2 FORMULATION:

The displacements satisfying all the boundary conditions can be expressed as

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\exp(i\omega t)$$
$$v(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\exp(i\omega t)$$
$$w(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\exp(i\omega t)$$

The transverse dynamic load in general can also be written as

$$q(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\exp(i\omega t) \tag{3.38}$$

Substituting the above displacement forms in (2.25), results in

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{Bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \end{Bmatrix}=\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \tag{3.39}$$

All the random quantities are expressed as

$$a_{ij}=a_{ij}^{d}+a_{ij}^{r} \quad \text{and} \quad q_j=q_j^{d}+q_j^{r} \quad \text{where i, j=1,2 and 3.}$$

Substituting all the random quantities in the equation (3.39), and collecting the zero order terms, we get

$$\begin{bmatrix} a_{11}^{d} & a_{12}^{d} & a_{13}^{d} \\ a_{21}^{d} & a_{22}^{d} & a_{23}^{d} \\ a_{31}^{d} & a_{32}^{d} & a_{33}^{d} \end{bmatrix}\begin{Bmatrix} \Lambda_1^{d} \\ \Lambda_2^{d} \\ \Lambda_3^{d} \end{Bmatrix}=\begin{Bmatrix} q_1^{d} \\ q_2^{d} \\ q_3^{d} \end{Bmatrix} \tag{3.40}$$

Solving the above deterministic equation (3.40) we can get the deterministic response.

Collecting the first order terms, we get

$$\begin{bmatrix} a_{11}{}^{d} & a_{12}{}^{d} & a_{13}{}^{d} \\ a_{21}{}^{d} & a_{22}{}^{d} & a_{23}{}^{d} \\ a_{31}{}^{d} & a_{32}{}^{d} & a_{33}{}^{d} \end{bmatrix} \begin{Bmatrix} \Lambda_1{}^{r} \\ \Lambda_2{}^{r} \\ \Lambda_3{}^{r} \end{Bmatrix} + \begin{bmatrix} a_{11}{}^{r} & a_{12}{}^{r} & a_{13}{}^{r} \\ a_{21}{}^{r} & a_{22}{}^{r} & a_{23}{}^{r} \\ a_{31}{}^{r} & a_{32}{}^{r} & a_{33}{}^{r} \end{bmatrix} \begin{Bmatrix} \Lambda_1{}^{d} \\ \Lambda_2{}^{d} \\ \Lambda_3{}^{d} \end{Bmatrix} = \begin{Bmatrix} q_1{}^{r} \\ q_2{}^{r} \\ q_3{}^{r} \end{Bmatrix} \tag{3.41}$$

Using Taylor's series expansion

$$a_{ij}{}^{r} = \sum_{l} \frac{\partial a_{ij}{}^{d}}{\partial b_l{}^{d}} b_l{}^{r}, \qquad \Lambda_j{}^{r} = \sum_{l} \frac{\partial \Lambda_j{}^{d}}{\partial b_l{}^{d}} b_l{}^{r} + \sum_{k} \frac{\partial \Lambda_j{}^{d}}{\partial q_k{}^{d}} q_k{}^{r} \tag{3.42}$$

l=1,2 and 3, i, j=1,2 and 3

Substituting the equation (3.42) in (3.41), the unknowns are $\frac{\partial \Lambda_j{}^{d}}{\partial b_l{}^{d}}$ and $\frac{\partial \Lambda_j{}^{d}}{\partial q_k{}^{d}}$. By solving equations (2.36) and (2.38), we can get the above said unknowns. Therefore, the total response becomes

$$\Lambda_j = \Lambda_j{}^{d} + \left[\sum_{l} b_l{}^{r} \frac{\partial \Lambda_j{}^{d}}{\partial b_l{}^{d}} + \sum_{k} q_k{}^{r} \frac{\partial \Lambda_j{}^{d}}{\partial q_k{}^{d}} \right] \tag{3.43}$$

The variance of the response can be evaluated as follows

$$\text{Var}\,(\Lambda_j) = E\left[\left(\Lambda_j - \Lambda_j{}^{d} \right)^2 \right]$$

$$\text{Var}\,(\Lambda_j{}^{r}) = E\left[\left(\sum_{l} b_l{}^{r} \frac{\partial \Lambda_j{}^{d}}{\partial b_l{}^{d}} + \sum_{k} q_k{}^{r} \frac{\partial \Lambda_j{}^{d}}{\partial q_k{}^{d}} \right)^2 \right]$$

$$= P^2 \sigma E_{11}{}^{r^2} + Q^2 \sigma E_{22}{}^{r^2} + R^2 \sigma U_{12}{}^{r^2} + S^2 \sigma q_2{}^{r^2} \tag{3.44}$$

where $P = \left(\frac{\partial \Lambda_j{}^{d}}{\partial E_{11}{}^{d}} \right)^2$, $q = \left(\frac{\partial \Lambda_j{}^{d}}{\partial E_{22}{}^{d}} \right)^2$, $R = \left(\frac{\partial \Lambda_j{}^{d}}{\partial U_{12}{}^{d}} \right)^2$ and $S = \left(\frac{\partial \Lambda_j{}^{d}}{\partial q_2{}^{d}} \right)^2$

$$SD(\Lambda^{r}) = \sqrt{P^2 \sigma E_{11}{}^{r^2} + Q^2 \sigma E_{22}{}^{r^2} + R^2 \sigma U_{12}{}^{r^2} + S^2 \sigma q_2{}^{r^2}} \tag{3.45}$$

i.e. Standard deviation of the response is the square root of variance of the response.

3.5 ANALYSIS OF HONEYCOMB

This section deals with the evaluation of equivalent material properties of honeycomb for stability and dynamic analysis of laminated sandwich panels. The In-plane equivalent core material properties of honeycomb have been defined as [23]

The equivalent modulus parallel to x,

$$E_1 = \frac{12E_s I\sigma_1 \cos\theta}{\sigma_1 (h + l\sin\theta) b l^2 \sin^2\theta} \tag{3.46}$$

The equivalent modulus parallel to y,

$$E_2 = \frac{12E_s I\sigma_2 (h + l\sin\theta)}{\sigma_2 b l^4 \cos^3\theta} \tag{3.47}$$

The Poisson's ratio's are calculated by

$$\upsilon_{12} = \frac{\cos^2\theta}{(\frac{h}{l} + \sin\theta)\sin\theta}, \qquad \upsilon_{21} = \frac{(\frac{h}{l} + \sin\theta)\sin\theta}{\cos^2\theta} \tag{3.48}$$

The shear modulus can be obtained as $G_{12} = \dfrac{12E_s I(h + l\sin\theta)}{lbh^2 \cos\theta(l + 2h)}$. (3.49)

For a cell with uniform thickness 't', the above properties reduces to

$$E_1 = (\frac{t}{l})^3 \frac{E_s \cos\theta}{(\frac{h}{l} + \sin\theta)\sin^2\theta} \qquad E_2 = (\frac{t}{l})^3 \frac{E_s(\frac{h}{l} + \sin\theta)}{\cos^3\theta}$$

$$G_{12} = (\frac{t}{l})^3 \frac{E_s(\frac{h}{l} + \sin\theta)}{(\frac{h}{l})^2 \cos\theta(1 + \frac{2h}{l})} \tag{3.50}$$

where E_1= Longitudinal modulus of honeycomb core,

E_2 = Transverse modulus of honeycomb core,

E_s= Modulus of elasticity of the solid from which the cell walls are made,

υ_{ij} = Poisson's ratios,

l = Inclined length of cell wall,

t = Cell wall thickness,

θ = Cell wall angle.

h = Height of side wall of cell

CHAPTER 4

RESULTS AND DISCUSSIONS

4.1 PRESENT WORK

Results have been obtained for cylindrical, spherical and conical shapes as particular cases of the general formulation in the previous chapter. Honeycomb sandwich laminated shell dimensions and lay up sequence are as shown in figure 4.1. Composite laminate is assumed to be axisymmetric with respect to shell mid surface. "B" matrix is zero due to symmetry. Moreover, the shell is assumed to be undergoing axisymmetric deformations, so the displacements and rotations are independent of the circumferential coordinate. Hence v_o drops from the formulation.

Mathematically this assumption of axisymmetric displacement can again be written as $\frac{\partial}{\partial y}(\ )=0$ and $v_o=0$. (4.1)

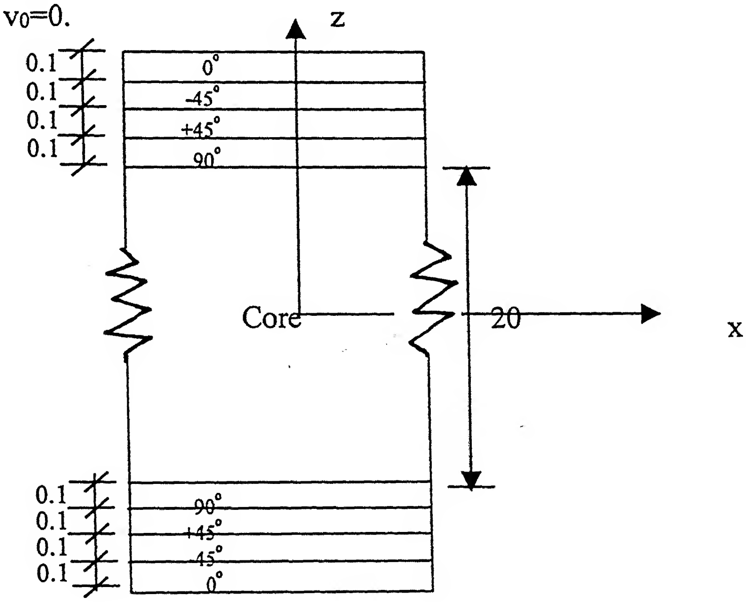


Figure 4.1: Lay up of the composite laminate (all the dimensions are in 'mm')

Substituting the above conditions (4.1) in the equations of motions of cylinder, sphere and conical shells the system coefficients take the following form for the specific shapes.

4.1.1 CYLINDRICAL SHELLS:

$$a_{11} = \frac{m^2\pi^2}{a^2}A_{11} \qquad a_{13} = -\frac{m\pi}{aR}A_{12} \qquad a_{31} = -\frac{m\pi}{aR}A_{12} \qquad a_{33} = \frac{A_{22}}{R^2} \tag{4.2}$$

4.1.2 SPHERICAL SHELLS:

$$a_{11} = \frac{m^2\pi^2}{a^2}A_{11} \qquad a_{13} = -\frac{m\pi}{aR}(A_{11} + A_{12})$$

$$a_{31} = -\frac{m\pi}{aR}(A_{11} + A_{12}) \qquad a_{33} = \frac{(A_{11} + 2A_{12} + A_{22})}{R^2} \tag{4.3}$$

4.1.3 CONICAL SHELLS:

$$a_{11} = \frac{m^2\pi^2}{a^2}A_{11} \qquad a_{13} = -\frac{m\pi}{aR_c}A_{12}$$

$$a_{31} = -\frac{m\pi}{aR_c}A_{12} \qquad a_{33} = \frac{A_{22}}{R_c^{\,2}} \tag{4.4}$$

Where R_c is the radius of the cone. It is a function of 'x' i.e. $R_c = R\ (x)$.

The sandwich shell is modeled as an equivalent laminated construction as discussed in chapter 3. The assumed mean values of input variables used for graphite-epoxy and equivalent properties of aluminum core are given in Table 1. Table 2 represents the dimensions of the shells.

Table 1: Characteristics of primary variables

Input Variable	Mean values of Graphite-epoxy	Mean values of Aluminum
E_{11}	3.05810 Gpa	0.00466 Gpa
E_{22}	0.06191 Gpa	0.00466 Gpa
υ_{12}	0.346	0.330
G_{12}	0.05088 Gpa	0.00129 Gpa
ρ	1.7gm/cc	0.0336gm/cc

Table 2: Dimensions of shells

Shell	Length (m)	Radius (m)
Cylinder	3.624	2.000
Sphere	1.176	1.176
Cone 1	2.920	2.00 - 1.176
Cone 2	0.980	2.00 - 1.400

4.2.BUCKLING DUE TO AXIAL COMPRESSION OF SHELLS:

The mean and the variance of the critical buckling load of cylinder, sphere and conical shells can be determined by solving equations (3.5) and (3.19), using equations (4.2), (4.3) and (4.4) respectively. Here '$N_x$cr' is the Critical buckling load in 'x' direction and 'a' is the length of the shell.

Numerical results are evaluated for the critical compressive load per unit circumference N_x for different types of shells considered

4.2.1 THE MEAN BUCKLING LOADS

Mean critical buckling load is determined from the equation (3.5) using equations (4.2), (4.3) and (4.4) for cylinder, sphere and conical shells respectively. The results for the mean of the critical buckling load (N_x) are presented in table 3

Table 3: Mean critical buckling loads for different shells

Shell	Critical buckling load (N/m)
Cylinder	318908.78125
Sphere	97129.26562
Cone1	207040.75000
Cone2	23320.73829

4.2.2 SD OF BUCKLING LOADS

Equation (3.20) gives the Standard Deviation of critical buckling load, N_x as a function of standard deviation of input random variables. The ratios of SD to mean for the material properties are taken as 0, 5, 10, 15 and 20 percent. The three inputs are varied independently and the results are presented five sets in tabular form.

Table 4 represents the change of SD/Mean of cylinder buckling load in five sub tables. Each table has one variation coefficient for υ_{12} with E_{11} and E_{22} variation coefficients assuming five values along the rows and the columns, respectively. Table 5 and 6 represent the corresponding values for the spherical and the conical shells.

It is inferred from the results that

- For the cone, the buckling load is position dependent as the radius changes along its length. The buckling load is observed to be increasing with decreasing radius of the shell due to increase of stiffness.
- All the shells are more strongly affected with changes in transverse modulus (E_{22}) than longitudinal modulus (E_{11}).

Table 4: SD/Mean of cylinder buckling load to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.0$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00000	0.01536	0.03073	0.04609	0.06146
0.05000	0.06592	0.06769	0.07273	0.08044	0.09013
0.10000	0.13185	0.13274	0.13538	0.13967	0.14547
0.15000	0.19777	0.19837	0.20015	0.20307	0.2071
0.20000	0.2637	0.26415	0.26548	0.2677	0.27076

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00007	0.01536	0.03073	0.04609	0.06146
0.05000	0.06146	0.06769	0.07273	0.08044	0.09013
0.10000	0.13185	0.13274	0.13538	0.13967	0.14547
0.15000	0.19777	0.19837	0.20015	0.20307	0.2071
0.20000	0.2637	0.26415	0.26548	0.2677	0.27076

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00015	0.01536	0.03073	0.04609	0.06146
0.05000	0.06592	0.06769	0.07273	0.08044	0.09013
0.10000	0.13185	0.13274	0.13538	0.13967	0.14547
0.15000	0.19777	0.19837	0.20015	0.20307	0.2071
0.20000	0.2637	0.26415	0.26548	0.2677	0.27076

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00022	0.01537	0.03073	0.04609	0.06146
0.05000	0.06592	0.06769	0.07273	0.08044	0.09013
0.10000	0.13185	0.13274	0.13538	0.13967	0.14547
0.15000	0.19777	0.19837	0.20015	0.20307	0.2071
0.20000	0.2637	0.26415	0.26548	0.2677	0.27076

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00030	0.01537	0.03073	0.04609	0.06146
0.05000	0.06592	0.06769	0.07273	0.08044	0.09013
0.10000	0.13185	0.13274	0.13538	0.13967	0.14547
0.15000	0.19777	0.19837	0.20015	0.20307	0.2071
0.20000	0.26370	0.26415	0.26548	0.2677	0.27077

sd/mean of $\upsilon_{12} = 0.20$

Table 5: SD/Mean of sphere buckling load to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.0$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00000	0.01536	0.03073	0.04609	0.06146
0.05000	0.06592	0.06769	0.07273	0.08044	0.09013
0.10000	0.13185	0.13185	0.13538	0.13967	0.14547
0.15000	0.19777	0.19837	0.20015	0.20307	0.20710
0.20000	0.26370	0.26415	0.26548	0.26770	0.27076

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.2000(
0.00000	0.00007	0.01536	0.03073	0.04609	0.0614€
0.05000	0.06592	0.06769	0.07273	0.08044	0.0901:
0.10000	0.13185	0.13185	0.13538	0.13967	0.14547
0.15000	0.19777	0.19837	0.20015	0.20307	0.20710
0.20000	0.26370	0.26415	0.26548	0.26770	0.27076

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00015	0.01537	0.03073	0.04609	0.06146
0.05000	0.06592	0.06769	0.07273	0.08044	0.09013
0.10000	0.13185	0.13185	0.13538	0.13967	0.14547
0.15000	0.19777	0.19837	0.20015	0.20307	0.20710
0.20000	0.26370	0.26415	0.26548	0.26770	0.27076

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00022	0.01537	0.03073	0.04609	0.06146
0.05000	0.06592	0.06769	0.07273	0.08044	0.09013
0.10000	0.13185	0.13274	0.13538	0.13967	0.14547
0.15000	0.19777	0.19837	0.20015	0.20307	0.20710
0.20000	0.26370	0.26415	0.26548	0.26770	0.27077

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00030	0.01537	0.03073	0.04609	0.06146
0.05000	0.06593	0.06769	0.07274	0.08044	0.09013
0.10000	0.13185	0.13274	0.13538	0.13967	0.14547
0.15000	0.19777	0.19837	0.20015	0.20307	0.20710
0.20000	0.26370	0.26415	0.26548	0.26707	0.27077

sd/mean of $\upsilon_{12} = 0.20$

Table 6: SD/Mean of cone buckling load to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.0$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00000	0.01536	0.03073	0.04609	0.06146
0.05000	0.06592	0.06769	0.07273	0.08044	0.09013
0.10000	0.13185	0.13185	0.13538	0.13967	0.14547
0.15000	0.19777	0.19837	0.20015	0.20307	0.20710
0.20000	0.26370	0.26415	0.26548	0.2677	0.27076

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00007	0.01536	0.03073	0.04609	0.06146
0.05000	0.06592	0.06769	0.07273	0.08044	0.09013
0.10000	0.13185	0.13185	0.13538	0.13967	0.14547
0.15000	0.19777	0.19837	0.20015	0.20307	0.20710
0.20000	0.26370	0.26415	0.26548	0.2677	0.27076

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00015	0.01537	0.03073	0.04609	0.06146
0.05000	0.06592	0.06769	0.07273	0.08044	0.09013
0.10000	0.13185	0.13185	0.13538	0.13967	0.14547
0.15000	0.19777	0.19837	0.20015	0.20307	0.20710
0.20000	0.26370	0.26415	0.26548	0.2677	0.27076

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00022	0.01537	0.03073	0.04609	0.06146
0.05000	0.06592	0.06769	0.07273	0.08044	0.09013
0.10000	0.13185	0.13274	0.13538	0.13967	0.14547
0.15000	0.19777	0.19837	0.20015	0.20307	0.20710
0.20000	0.26370	0.26415	0.26548	0.2677	0.27077

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.0003	0.01537	0.03073	0.04609	0.06146
0.05000	0.06593	0.06769	0.07274	0.08044	0.09013
0.10000	0.13185	0.13274	0.13538	0.13967	0.14547
0.15000	0.19777	0.19837	0.20015	0.20307	0.20710
0.20000	0.26370	0.26415	0.26548	0.2677	0.27077

sd/mean of $\upsilon_{12} = 0.20$

- All the shells are least affected with changes in the Poisons ratio υ_{12}.
- The SD/Mean of buckling load is linearly varying with the SD/Mean of the input random variables considered.

4.3. FREE VIBRATION OF SHELLS:

The natural frequencies of cylinder, sphere and conical shells have been determined by setting the external loads and in-plane loads as zeros in equation (2.10) and by solving equation (3.27) using (4.2), (4.3) and (4.4) respectively. The Variance and Standard deviation of ω^2 have been determined by equations (3.36) and (3.37). As mentioned earlier the deflection is assumed to be axisymmetric.

4.3.1 MEAN NATURAL FREQUENCIES

In this case, the characteristic equation, (equation (3.27)), for the mean of natural frequency is quadratic. Hence, there are two roots corresponding to the value of λ. Table7 gives the mean of first two natural frequencies of cylinder, sphere and conical shells corresponding to first five wave numbers..

Table7: Mean of Natural Frequencies (HZ)

Mode	Frequency	Cylinder	Sphere	Cone1	Cone2
1	1^{st}	335.7443	535.4978	339.02500	343.13251
	2^{nd}	643.2385	2113.436	790.5953	2327.45337
2	1^{st}	342.0127	571.5809	342.5817	343.45821
	2^{nd}	1262.898	3960.036	1564.7740	4650.49658
3	1^{st}	342.9012	578.6065	343.1382	343.51700
	2^{nd}	1889.439	5867.921	2343.354	6974.54346
4	1^{st}	343.1963	581.0881	343.3264	343.53992
	2^{nd}	2517.086	7790.495	3122.758	9298.83203
5	1^{st}	343.3300	582.2404	343.4131	343.54565
	2^{nd}	3145.131	9718.854	3902.469	11623.21680

4.3.2 SD OF NATURAL FREQUENCIES

The SD of the natural frequencies is obtained as a function of SD of input random variables from the equation (3.37). The ratio of SD to mean for the material properties is varied as mentioned earlier for the buckling study.

The Table (8) to table (22) represents the change of SD/Mean of square of natural frequencies with respect to the SD/Mean of random variables for all shells up to mode 5. The normalized results for cone1 and cone2 are identical. The following conclusions can be made from the results

- Cylinder and spherical shells are more strongly affected with changes in longitudinal modulus (E_{11}) than transverse modulus (E_{22}).
- Conical shells show greater sensitivity with changes in transverse modulus (E_{22}) than longitudinal modulus (E_{11}).
- All the shells are least affected with changes in the Poisons ratio υ_{12}.
- As the mode number increases, the influence of E_{11} is decreasing and that of E_{22} is increasing for all the shells.
- The SD/Mean of square of the natural frequency for cone1 and cone2 is same due to the normalization scheme adopted.
- The SD/Mean of square of the natural frequency increases linearly with the input SD/Mean of random variables for all the shells.
- The stiffness of conical shell is position dependent as the radius changes along its length. The natural frequencies are observed to be increasing with decreasing radius of the shell.
- The natural frequencies increase with increase in the mode number.

Table (8): SD/Mean of square of natural frequency of cylinder for model1 to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00029	0.20598	0.41195	0.61793	0.82391
0.05000	0.06761	0.21679	0.41746	0.62162	0.82668
0.10000	0.13521	0.24639	0.43557	0.63255	0.83493
0.15000	0.20282	0.28907	0.45917	0.65036	0.84850
0.20000	0.27042	0.33993	0.49278	0.67451	0.86715

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00058	0.20598	0.41195	0.61793	0.82391
0.05000	0.06761	0.21679	0.41746	0.62162	0.82668
0.10000	0.13521	0.24639	0.43557	0.63255	0.83493
0.15000	0.20282	0.28907	0.45917	0.65036	0.84850
0.20000	0.27042	0.33993	0.49278	0.67451	0.86715

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00087	0.20598	0.41195	0.61793	0.82391
0.05000	0.06761	0.21679	0.41746	0.62162	0.82668
0.10000	0.13521	0.24639	0.43557	0.63255	0.83493
0.15000	0.20282	0.28907	0.45917	0.65036	0.84850
0.20000	0.27042	0.33993	0.49278	0.67451	0.86715

Table (9): SD/Mean of square of natural frequency of cylinder for mode2 to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00014	0.01170	0.02340	0.03510	0.04680
0.05000	0.07276	0.07370	0.07643	0.08079	0.08651
0.10000	0.14552	0.14599	0.14739	0.14970	0.15286
0.15000	0.21829	0.21860	0.21954	0.22109	0.22325
0.20000	0.29105	0.29128	0.29199	0.29316	0.29479

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00029	0.01170	0.02340	0.03510	0.04680
0.05000	0.07276	0.07370	0.07643	0.08079	0.08651
0.10000	0.14552	0.14599	0.14739	0.14970	0.15286
0.15000	0.21829	0.21860	0.21954	0.22109	0.22325
0.20000	0.29105	0.29128	0.29199	0.29316	0.29479

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00043	0.01170	0.02340	0.03510	0.04680
0.05000	0.07276	0.07370	0.07643	0.08079	0.08651
0.10000	0.14552	0.14599	0.14739	0.14970	0.15286
0.15000	0.21829	0.21860	0.21954	0.22109	0.22325
0.20000	0.29105	0.29128	0.29199	0.29316	0.29479

Table (10): SD/Mean of square of natural frequency of cylinder for mode 3 to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00021	0.03996	0.07992	0.11988	0.15984
0.05000	0.07344	0.08361	0.10854	0.14059	0.17590
0.10000	0.14687	0.15221	0.16721	0.18959	0.21707
0.15000	0.22031	0.22391	0.23436	0.25082	0.27219
0.20000	0.29375	0.29645	0.30443	0.31727	0.33442

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00042	0.03996	0.07992	0.11988	0.15984
0.05000	0.07344	0.08361	0.10854	0.14059	0.17590
0.10000	0.14687	0.15221	0.16721	0.18959	0.21707
0.15000	0.22031	0.22391	0.23436	0.25082	0.27219
0.20000	0.29375	0.29645	0.30443	0.31727	0.33442

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00063	0.03996	0.07992	0.11988	0.15984
0.05000	0.07344	0.08361	0.10854	0.14059	0.17590
0.10000	0.14687	0.15221	0.16721	0.18959	0.21707
0.15000	0.22031	0.22391	0.23436	0.25082	0.27219
0.20000	0.29375	0.29645	0.30443	0.31727	0.33442

Table (11): SD/Mean of square of natural frequency of cylinder for mode 4 to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00023	0.04918	0.09836	0.14753	0.19671
0.05000	0.07366	0.08857	0.12288	0.16490	0.21005
0.10000	0.14732	0.15531	0.17713	0.20849	0.24756
0.15000	0.22097	0.22638	0.24187	0.26570	0.29585
0.20000	0.29463	0.29871	0.31062	0.32951	0.35427

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00047	0.04918	0.09836	0.14753	0.19671
0.05000	0.07366	0.08857	0.12288	0.16490	0.21005
0.10000	0.14732	0.15531	0.17713	0.20849	0.24756
0.15000	0.22097	0.22638	0.24187	0.26570	0.29585
0.20000	0.29463	0.29871	0.31062	0.32951	0.35427

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00070	0.04918	0.09836	0.14753	0.19671
0.05000	0.07366	0.08857	0.12288	0.16490	0.21005
0.10000	0.14732	0.15531	0.17713	0.20849	0.24756
0.15000	0.22097	0.22638	0.24187	0.26570	0.29585
0.20000	0.29463	0.29871	0.31062	0.32951	0.35427

Table (12): SD/Mean of square of natural frequency of cylinder for mode 5 to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00024	0.05333	0.10666	0.15998	0.21331
0.05000	0.07376	0.09102	0.12967	0.17617	0.22570
0.10000	0.14752	0.15686	0.18203	0.21761	0.25935
0.15000	0.22127	0.22761	0.24564	0.27305	0.30735
0.20000	0.29503	0.29891	0.31372	0.33562	0.36407

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00049	0.05333	0.10666	0.15998	0.21331
0.05000	0.07376	0.09102	0.12967	0.17617	0.22570
0.10000	0.14752	0.15686	0.18203	0.21761	0.25935
0.15000	0.22127	0.22761	0.24564	0.27305	0.30735
0.20000	0.29503	0.29891	0.31372	0.33562	0.36407

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00073	0.05333	0.10666	0.15998	0.21331
0.05000	0.07376	0.09102	0.12967	0.17617	0.22570
0.10000	0.14752	0.15686	0.18203	0.21761	0.25935
0.15000	0.22127	0.22761	0.24564	0.27305	0.30735
0.20000	0.29503	0.29891	0.31372	0.33562	0.36407

Table (13): SD/Mean of square of natural frequency of sphere for mode 1 to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00187	0.29894	0.59786	0.89679	1.19572
0.05000	0.05985	0.30486	0.60085	0.89879	1.19722
0.10000	0.11966	0.32199	0.60972	0.90474	1.20170
0.15000	0.17948	0.34867	0.62422	0.91458	1.20912
0.20000	0.23931	0.38292	0.64398	0.92817	1.21943

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00373	0.29895	0.58789	0.89880	1.19573
0.05000	0.05994	0.30488	0.60086	0.89879	1.19722
0.10000	0.11971	0.32201	0.60973	0.90475	1.20170
0.15000	0.17951	0.34869	0.62423	0.91458	1.20912
0.20000	0.23933	0.38293	0.64398	0.92818	1.21944

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00560	0.29898	0.58791	0.89881	1.19574
0.05000	0.06009	0.30491	0.60087	0.89880	1.19723
0.10000	0.11978	0.32204	0.60974	0.90476	1.20171
0.15000	0.17956	0.34871	0.62424	0.91459	1.20913
0.20000	0.23936	0.38296	0.64400	0.92819	1.21945

Table (14): SD/Mean of square of natural frequency of sphere for mode 2 to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00178	0.08515	0.17026	0.25339	0.34051
0.05000	0.06440	0.10674	0.18203	0.26337	0.34654
0.10000	0.12876	0.15435	0.21346	0.28600	0.36404
0.15000	0.19313	0.21106	0.25746	0.32018	0.39146
0.20000	0.25750	0.27120	0.30869	0.36266	0.42691

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00356	0.08520	0.17029	0.25541	0.34053
0.05000	0.06447	0.10679	0.18205	0.26339	0.34656
0.10000	0.12880	0.15439	0.21348	0.28602	0.36405
0.15000	.019315	0.21108	0.25748	0.32020	0.39148
0.20000	0.25752	0.27122	0.30871	0.36268	0.42692

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00535	0.08529	0.17034	0.25544	0.34055
0.05000	0.06459	0.10686	0.18210	0.26342	0.34658
0.10000	0.12886	0.15444	0.21352	0.28605	0.36407
0.15000	0.19319	0.21112	0.25751	0.32022	0.39150
0.20000	0.25755	0.27125	0.30873	0.36270	0.42694

Table (15): SD/Mean of square of natural frequency of sphere for mode 3 to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00176	0.04614	0.09222	0.13832	0.18442
0.05000	0.06523	0.07998	0.11295	0.15292	0.19561
0.10000	0.13043	0.13834	0.15973	0.19011	0.22588
0.15000	0.19564	0.20100	0.21628	0.23959	0.26886
0.20000	0.26085	0.26489	0.27667	0.29525	0.31945

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00352	0.04624	0.02928	0.13836	0.18445
0.05000	0.06531	0.07994	0.11299	0.15295	0.19564
0.10000	0.13047	0.13837	0.15976	0.19014	0.22590
0.15000	0.19566	0.20102	0.21630	0.23961	0.26887
0.20000	0.26087	0.26491	0.27668	0.29526	0.31947

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00529	0.04641	0.09236	0.13841	0.18449
0.05000	0.06542	0.08004	0.11306	0.15300	0.19568
0.10000	0.13053	0.13843	0.15981	0.19018	0.22594
0.15000	0.19570	0.20106	0.21634	0.23964	0.26890
0.20000	0.26089	0.26494	0.27671	0.29529	0.31949

Table (16): SD/Mean of square of natural frequency of sphere for mode 4 to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00176	0.03263	0.06519	0.09776	0.13034
0.05000	0.06550	0.07318	0.09241	0.11767	0.14587
0.10000	0.13101	0.13500	0.14632	0.16346	0.18479
0.15000	0.19653	0.19919	0.20703	0.21948	0.23580
0.20000	0.26201	0.26403	0.26999	0.27965	0.29263

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00351	0.03277	0.06526	0.09781	0.13037
0.05000	0.06552	0.07324	0.09246	0.11771	0.14590
0.10000	0.13105	0.13504	0.14635	0.16348	0.18482
0.15000	0.19657	0.19921	0.20705	0.21950	0.23582
0.20000	0.26203	0.26404	0.27001	0.27966	0.29265

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00527	0.03300	0.06537	0.09788	0.13043
0.05000	0.06559	0.07335	0.09254	0.11778	0.14595
0.10000	0.13111	0.13509	0.14641	0.16353	0.18486
0.15000	0.19663	0.19925	0.20709	0.21953	0.23585
0.20000	0.26205	0.26407	0.27003	0.27969	0.29267

Table (17): SD/Mean of square of natural frequency of sphere for mode 5 to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00175	0.02643	0.05278	0.07915	0.10552
0.05000	0.06566	0.07076	0.08423	0.10282	0.12427
0.10000	0.13128	0.13390	0.14148	0.15328	0.16842
0.15000	0.19691	0.19867	0.20386	0.21222	0.22340
0.20000	0.26254	0.26387	0.26779	0.27421	0.28295

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00350	0.02661	0.05287	0.07921	0.10556
0.05000	0.06573	0.07324	0.08428	0.10287	0.12430
0.10000	0.13132	0.13394	0.14152	0.15331	0.16845
0.15000	0.19694	0.19869	0.20388	0.21224	0.22342
0.20000	0.26256	0.26388	0.26781	0.27423	0.28297

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00525	0.02689	0.05301	0.07930	0.10564
0.05000	0.06584	0.07335	0.08437	0.10294	0.12437
0.10000	0.13137	0.13400	0.14157	0.15336	0.16850
0.15000	0.19697	0.19873	0.20392	0.21227	0.22345
0.20000	0.26259	0.26391	0.26784	0.27426	0.28299

Table (19): SD/Mean of square of natural frequency of cone for mode 2 to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00001	0.05305	0.10611	0.15916	0.21221
0.05000	0.06506	0.08395	0.12446	0.17194	0.22196
0.10000	0.13012	0.14052	0.16790	0.20558	0.24893
0.15000	0.19518	0.20226	0.22216	0.25185	0.28832
0.20000	0.26024	0.26560	0.28104	0.30505	0.33580

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00001	0.05305	0.10611	0.15916	0.21221
0.05000	0.06506	0.08395	0.12446	0.17194	0.22196
0.10000	0.13012	0.14052	0.16790	0.20558	0.24893
0.15000	0.19518	0.20226	0.22216	0.25185	0.28832
0.20000	0.26024	0.26560	0.28104	0.30505	0.33580

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00002	0.05305	0.10611	0.15916	0.21221
0.05000	0.06506	0.08395	0.12446	0.17194	0.22196
0.10000	0.13012	0.14052	0.16790	0.20558	0.24893
0.15000	0.19518	0.20226	0.22216	0.25185	0.28832
0.20000	0.26024	0.26560	0.28104	0.30505	0.33580

Table (20): SD/Mean of square of natural frequency of cone for mode 3 to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00004	0.03151	0.06302	0.09452	0.12603
0.05000	0.06552	0.07271	0.09091	0.11501	0.14205
0.10000	0.13105	0.13478	0.14541	0.16158	0.18182
0.15000	0.19657	0.19908	0.20643	0.21812	0.23350
0.20000	0.26210	0.26398	0.26956	0.27862	0.29082

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00008	0.03151	0.06302	0.09452	0.12603
0.05000	0.06552	0.07271	0.09091	0.11501	0.14205
0.10000	0.13105	0.13478	0.14541	0.16158	0.18182
0.15000	0.19657	0.19908	0.20643	0.21812	0.23350
0.20000	0.26210	0.26398	0.26956	0.27862	0.29082

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00012	0.03151	0.06302	0.09452	0.12603
0.05000	0.06552	0.07271	0.09091	0.11501	0.14205
0.10000	0.13105	0.13478	0.14541	0.16158	0.18182
0.15000	0.19657	0.19908	0.20643	0.21812	0.23350
0.20000	0.26210	0.26398	0.26956	0.27862	0.29082

Table (21): SD/Mean of square of natural frequency of cone for mode 4 to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00005	0.02433	0.04865	0.07298	0.09730
0.05000	0.06568	0.07004	0.08173	0.09818	0.11739
0.10000	0.13136	0.13359	0.14008	0.15027	0.16347
0.15000	0.19704	0.19853	0.20295	0.21012	0.21975
0.20000	0.26271	0.26384	0.26718	0.27266	0.28015

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00011	0.02433	0.04865	0.07298	0.09730
0.05000	0.06568	0.07004	0.08173	0.09818	0.11739
0.10000	0.13136	0.13359	0.14008	0.15027	0.16347
0.15000	0.19704	0.19853	0.20295	0.21012	0.21975
0.20000	0.26271	0.26384	0.26718	0.27266	0.28015

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00016	0.02433	0.04865	0.07298	0.09730
0.05000	0.06568	0.07004	0.08173	0.09818	0.11739
0.10000	0.13136	0.13359	0.14008	0.15027	0.16347
0.15000	0.19704	0.19853	0.20295	0.21012	0.21975
0.20000	0.26271	0.26384	0.26718	0.27266	0.28015

Table (22): SD/Mean of square of natural frequency of cone for mode 5 to SD/Mean of all RV's

sd/mean of $\upsilon_{12} = 0.05$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00006	0.02106	0.04213	0.06319	0.08426
0.05000	0.06575	0.06904	0.07809	0.09119	0.10688
0.10000	0.13150	0.13317	0.13808	0.14589	0.15618
0.15000	0.19725	0.19837	0.20170	0.20712	0.21449
0.20000	0.26299	0.26384	0.26635	0.27048	0.27616

sd/mean of $\upsilon_{12} = 0.10$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00012	0.02106	0.04213	0.06319	0.08426
0.05000	0.06575	0.06904	0.07809	0.09119	0.10688
0.10000	0.13150	0.13317	0.13808	0.14589	0.15618
0.15000	0.19725	0.19837	0.20170	0.20712	0.21449
0.20000	0.26299	0.26384	0.26635	0.27048	0.27616

sd/mean of $\upsilon_{12} = 0.15$

sd/mean E_{22} \ E_{11}	0.00000	0.05000	0.10000	0.15000	0.20000
0.00000	0.00018	0.02106	0.04213	0.06319	0.08426
0.05000	0.06575	0.06904	0.07809	0.09119	0.10688
0.10000	0.13150	0.13317	0.13808	0.14589	0.15618
0.15000	0.19725	0.19837	0.20170	0.20712	0.21449
0.20000	0.26299	0.26384	0.26635	0.27048	0.27616

4.4 FORCED VIBRATION OF SHELLS:

The deflection amplitudes of cylinder, sphere and conical shells have been determined by solving the equation (3.40). The Variance and Standard deviation of deflections are determined from the equations (3.44) and (3.45).

Numerical results are presented for the forced vibrations of cylinder, sphere and conical shells. The boundary conditions for the shell are considered to be simply supported. The deflections, as mentioned earlier are assumed to be axisymmetric. All the SD of response is normalized with their respective mean values. The assumed mean values of the input primary variables are given in Table1. The mean external load is taken as 5000 N/sq.mm.

4.4.1 MEAN RESPONSE AMPLITUDE

The mean response can be determined by solving equation (3.40). Table (23) gives the mean response of cylinder, sphere and conical shells.

Table (23): Mean Deflection amplitude

Shell	U - Deflection (m)	W - Deflection (m)
Cylinder	0.00387	0.02103
Sphere	0.00303	0.00723
Cone1	0.00311	0.02103
Cone2	0.00104	0.02102

4.4.2 SD OF RESPONSE

The SD of the deflection amplitude is obtained as a function of SD of input random variables from the equation (3.45). The ratio of SD to mean for the material properties and

the excitation are considered to change independently. The results are obtained for their values as 0, 5, 10, 15 and 20 percent.

The Figures 4.2 to 4.5 represent the change of SD/Mean of response with respect to the SD/Mean of input random variables.

It is observed from the results that

- All the shells are more strongly affected with changes in longitudinal modulus (E_{11}) and the external loading (Q) compared to changes in transverse modulus (E_{22}) for deflection - U
- All the shells are more sensitive to changes in transverse modulus (E_{22}) and external loading (Q) than the longitudinal modulus (E_{11}) for deflection - W.
- The mean response is decreasing with decreasing of radius of conical shell due to change in stiffness along its length.
- All the shells are least affected with changes in the Poison's ratio υ_{12}.
- The SD/Mean of deflection for cone1 and cone2 is same due to normalization.
- The SD/Mean of response increases linearly with the input SD/Mean of random variables for all the shells.
- The SD/Mean of deflections increases with increase in mode number for all the shells.

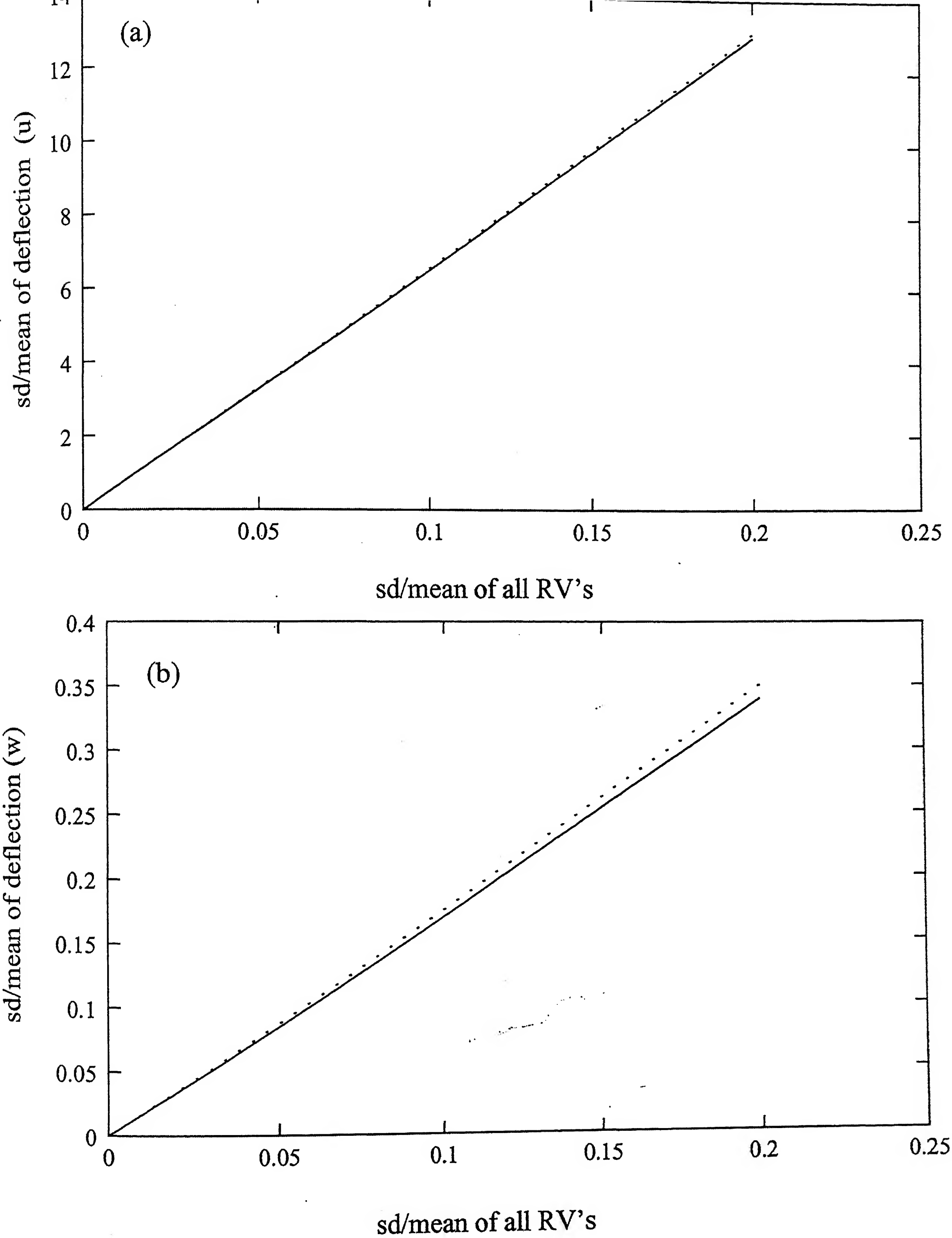
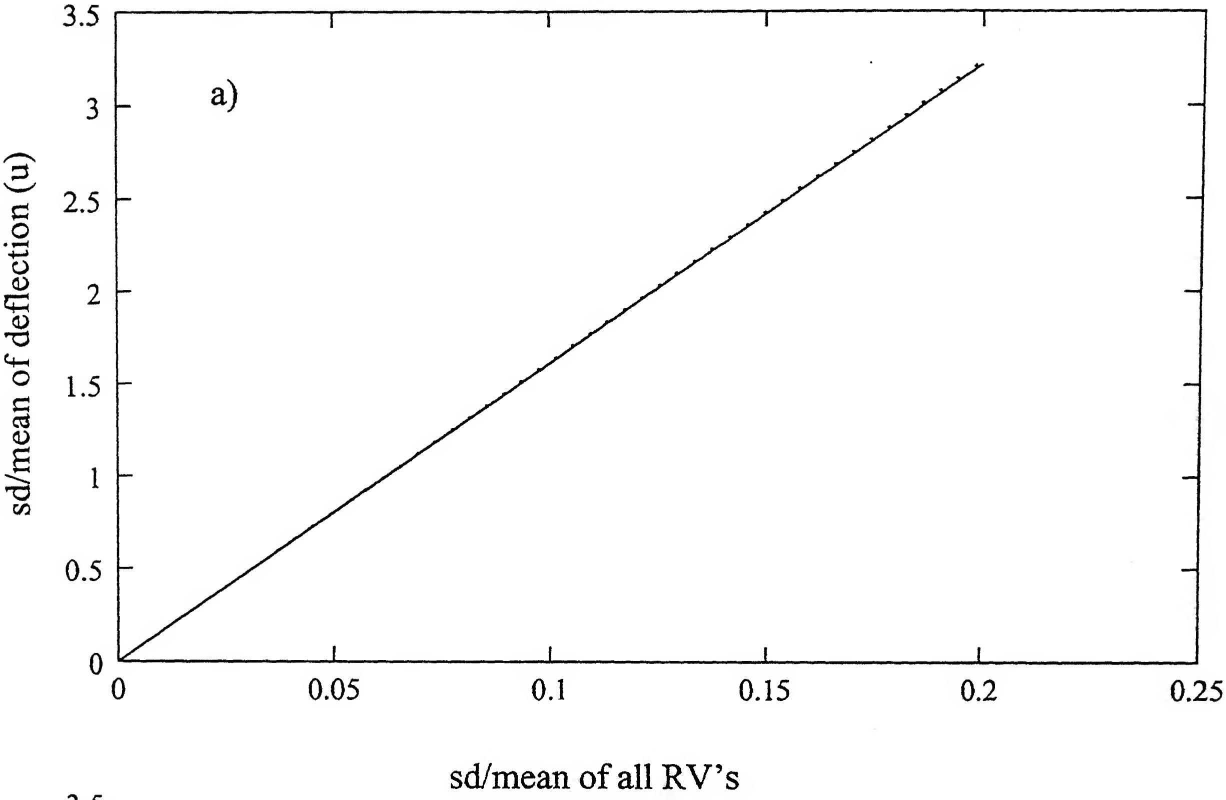


Figure 4.2: Variation of sd/mean of deflection for cylindrical panel
(a) u – deflection (b) w – deflection



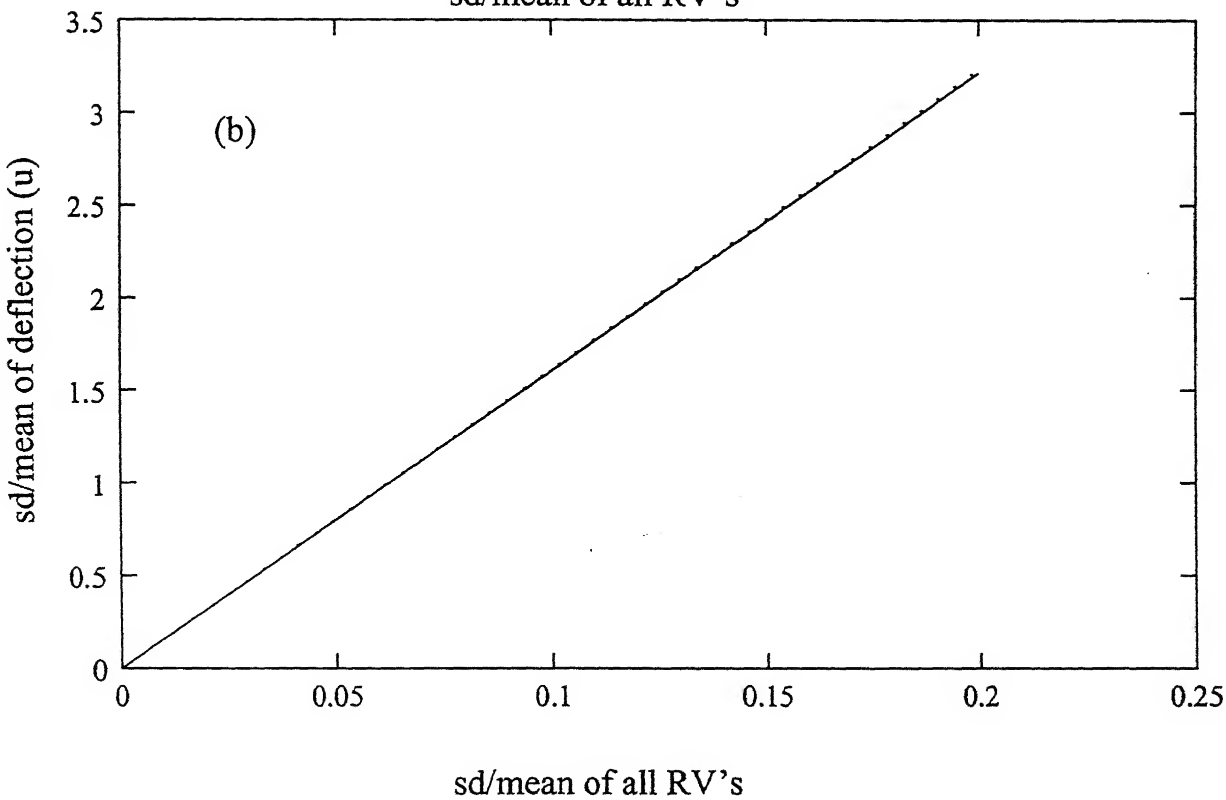
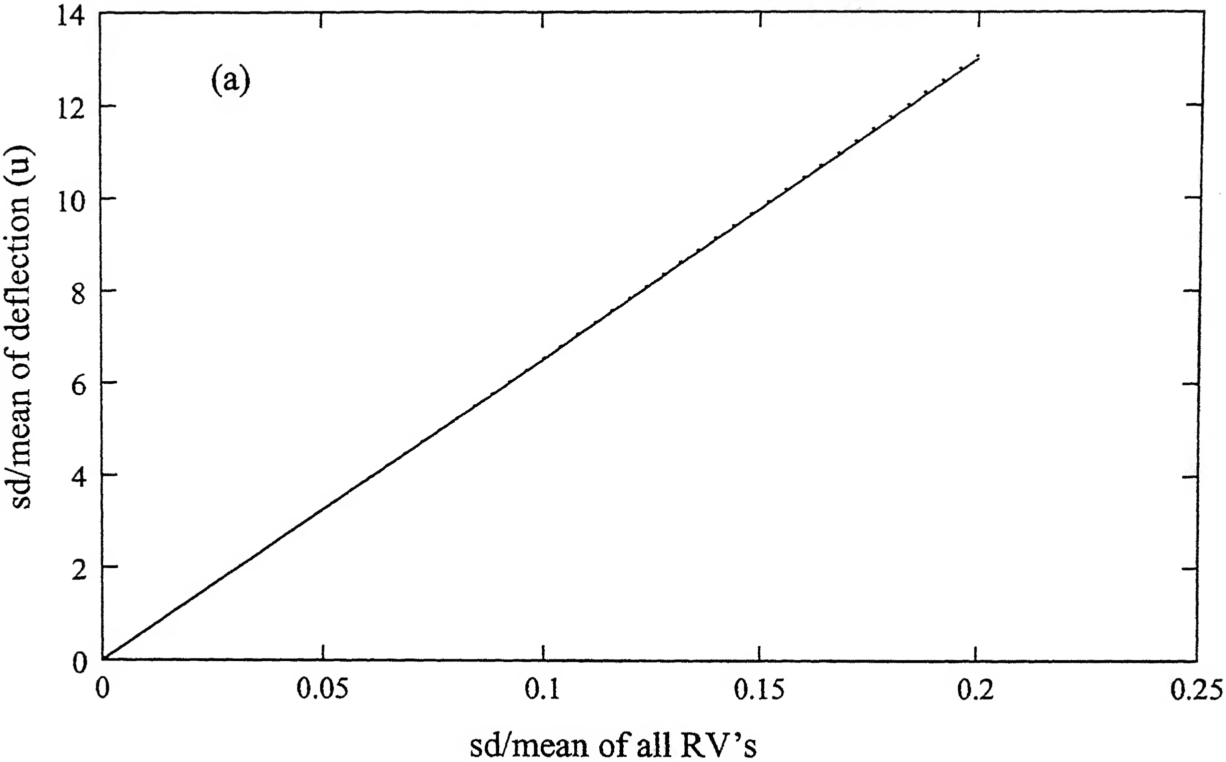


Figure 4.3: Variation of sd/mean of deflections for spherical panel
(a) u – deflection (b) w – deflection



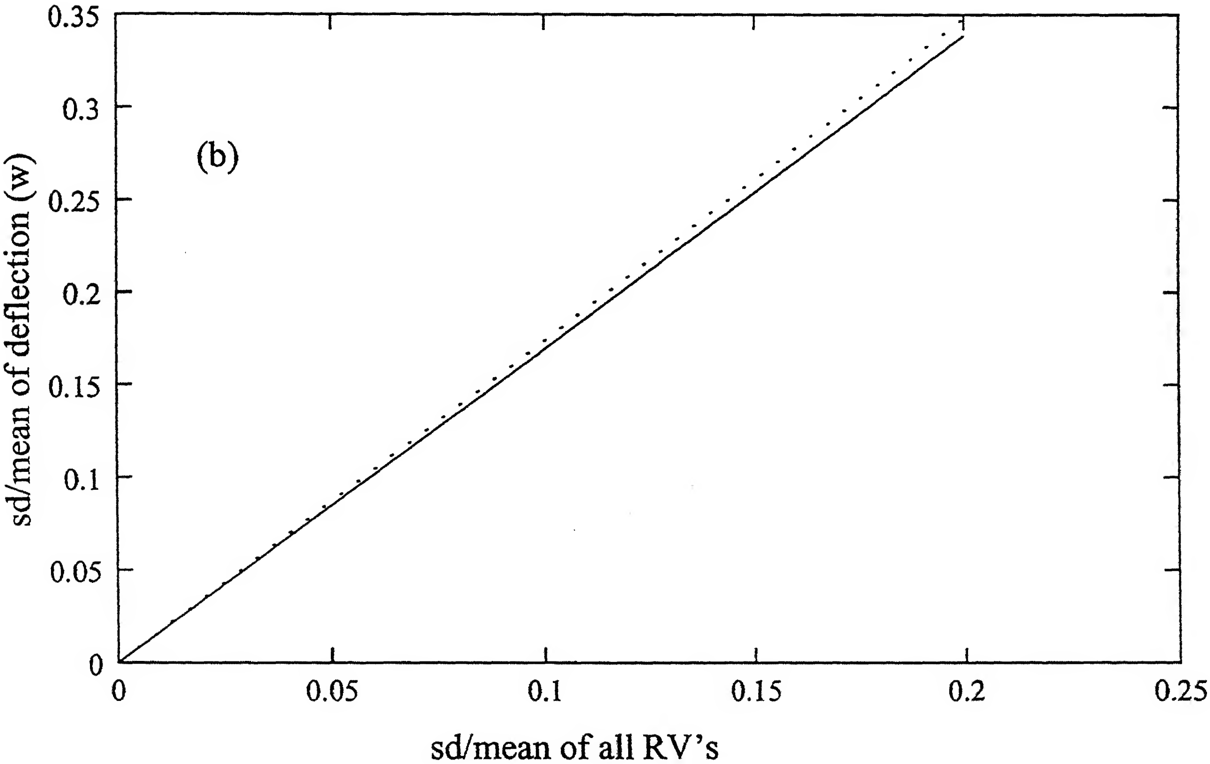


Figure 4.4: Variation of sd/mean of conical panel−1
(a) u − deflection (b) w − deflection

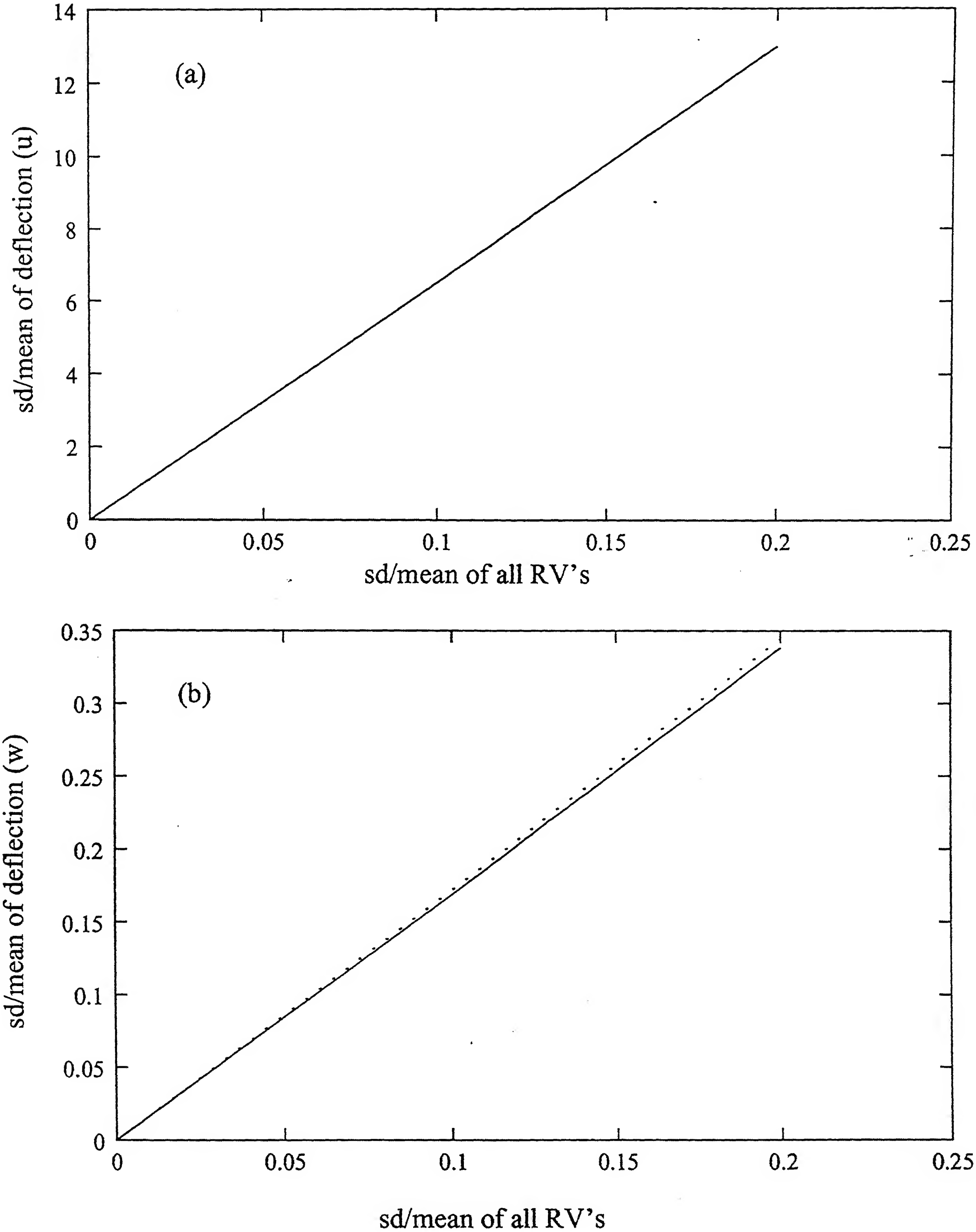


Figure 4.5: Variation of sd/mean of deflections for conical panel 2
(a) u – deflection (b) w – deflection

4.5 SATELLITE PAYLOAD FAIRING:

A typical satellite payload fairing considered here consists of 3 types of shells namely cylinder, sphere and cone of different dimensions (Figure 4.6). Results for independent stability and dynamic analysis of various shells have been presented in this chapter. But in the complete structure, these component shells are attached together and support each other.

For the satellite payload fairing, two extremes of ideal support conditions may be considered for the shells to be simply supported and fixed ends. Simply supported condition allows rotation but not deflection but the fixed condition neither allows rotation nor deflection. The satellite payload fairing shell components will actually have flexible support conditions such that they allow some rotation and also some deflection.

It can be conclude from the results that,

- The critical buckling load of the entire structure for the assumed support conditions is the lowest among all the critical buckling loads of shell given in table 3, which is for cone2.
- The lowest natural frequency of the entire system for the assumed support conditions is the lowest among all the frequencies presented in the table 7, which is for cylinder.
- The maximum deflections along x and z-axis respectively are the largest values among all the deflections presented in the table 23, which is for cylinder.
- The critical buckling load, natural frequency and the deflections depends on the support conditions, dimensions of the shell, material properties and external load also.

- For the actual conditions, the behavior of satellite payload fairing will be in between the behavior of the structure when the support conditions are simply supported and fixed.

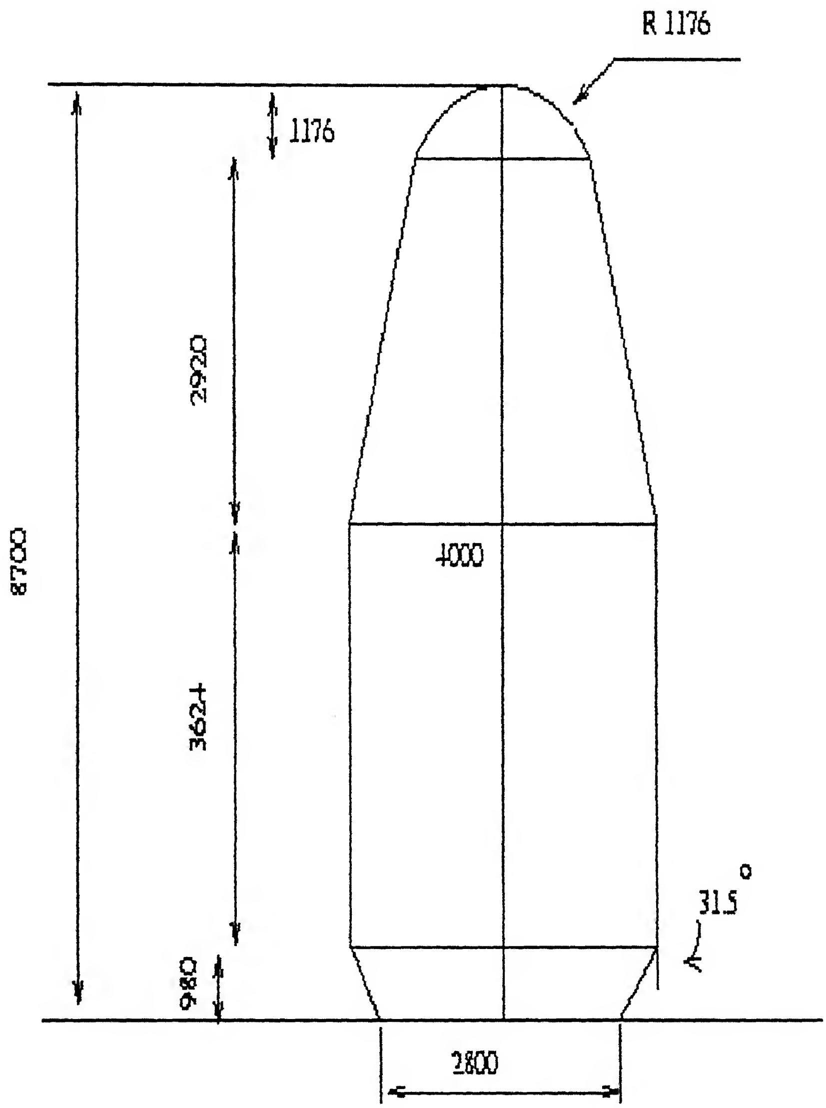


Figure 4.6: Typical satellite payload fairing

CHAPTER 5

CONCLUSIONS AND SCOPE FOR FUTURE WORK

5.1 CONCLUSIONS

A general approach has been presented for stability and dynamic analysis of laminated sandwich shells with random material properties and external loading. The second order statistics, that is, mean and standard deviation of buckling load, natural frequencies and forced response amplitude characteristics have been obtained for cylinder, sphere and conical shells as a function of known statistics of input random variables using perturbation technique. The method has been utilized to obtain numerical results for a specific shell configuration. In the buckling analysis, the buckling of shells under axial compressive load is studied. Axisymmetric vibration modes are considered in the vibrational analysis. Some conclusions that are based on this study are as follows:

- SD/Mean of buckling load, natural frequency and deflection changes linearly with SD/Mean of input RV's.
- Larger response SD is associated with higher mean response. Hence for higher magnitude of loads, random analysis is more important for accurate evaluation of laminated shell response behaviour.
- All the shells are least affected with changes in the Poison's ratio υ_{12}.
- The buckling loads and natural frequencies show different sensitivity to randomness in different basic material properties. The dominant material

property depends on the laminate construction, mode of deflection/vibration, and geometrical parameters.

- The buckling load increases with decreasing radius of the shell.
- All the shells in buckling are more strongly affected with changes in transverse modulus (E_{22}) than longitudinal modulus (E_{11}).
- For the free vibrations, Cylinder and spherical shells are strongly affected with changes in longitudinal modulus (E_{11}) and Conical shells show greater sensitivity with changes in transverse modulus (E_{22}).
- The effect of E_{11} decreases and that of E_{22} increases with the increase in mode number in free vibration.
- The ratio of SD/Mean of buckling load is same for cylinder, sphere and conical shell due to the normalization scheme adopted.
- In forced vibration, the longitudinal deflection – u, all the shells are strongly affected with changes in E_{11} and Q and for radial deflection – w, all the shells are sensitive to the changes in E_{22} and Q.

5.2 SCOPE FOR FUTURE WORK

The present study is confined to analysis of thin shells ($h/R<0.1$) so that classical thin shell theory is used. In this study, transverse shear deformation effects and rotatory inertia effects are neglected. The present study can be extended to analysis of thick shells including transverse shear deformation and rotatory inertia effects with different support conditions to develope a more complete understanding of composite shell behaviour.

The first order perturbation terms of random variables are only considered in the present study and higher order terms are neglected in comparison with their corresponding mean values. For future work, second and higher order terms can be considered to make the analysis more precise and valid in the nonlinear domain.

REFERENCES

[1] N.C.Nigam and S.Narayanan, "Applications of Random Vibrations", Narosa Publishing House, New Delhi, (1994).

[2] S.H.Crandall, "Random Vibration", John Wiley & Sons, Inc., New York, (1958).

[3] Y.K.Lin, "Probabilistic theory of structural dynamics", McGraw-Hill, New York, (1967).

[4] R.A.Ibrahim, "Structural dynamics with parameter uncertainties", Trans. ASME, Applied mechanics review, vol.40. No.3, (1987) PP.309-328.

[5] C.S.Manohar and R.A.Ibrahim, "Progress in structural dynamics with stochastic parameter variations": 1997-1998, Trans.ASME, Applied Mechanic Review, 52(5) (1999), pp.177-196.

[6] R.Vaicities, "Free Vibrations of beams with random characteristics", Journal of Sound and Vibration, 35(1), 1975. pp 13.

[7] P.C.Chen and W.W. Soroka, "Multi-degree dynamic response of a system with statistical properties", J.Sound and Vibration, 37(4), 1974, pp. 547-56.

[8] J.R.Vinson and R.L.Sierakowski, " The behaviour of structures composed of composite materials", Martinus Nijhoff Publishers, Dordrecht, (1986).

[9] A.W.Leissa and A.F.Martin, "Vibration and buckling of rectangular composite plates with variable fiber spacing", Composite Structures, 14, 1990, pp.339-357.

[10] Z.Zhang and S.Chen, "The standard deviations of the eigen solutions for random MDOF systems", Computers and Structures, 39(6), 1991, pp.603-607.

[11] S.Salim, D.Yadav, and N.G.R.Iyengar, "Analysis of composite plates with random material characteristics", Mechanics Research Communication, vol.20, no.5, (1993) pp.405-414.

[12] S.Salim, D.Yadav and N.G.R.Iyengar, "Free vibration of composite plates with random material characteristics", 1993.

[13] B.N.Singh, "probabilistic response of laminated composite panels with random material properties", PhD thesis, Department of Aerospace Engineering, IIT KANPUR, August 2001.

[14] Shinozuka, M. and Astill, C.J., "Random Eigen Value problems in structure analysis", American Institute of Aeronautics and Astronautics Journal 10(4), 1972, 456-462.

[15] I.Elishakoff, "Probabilistic methods in theory of structures", John Wiley & Sons, New York, (1983).

[16] S.Adali, I.Elishakoff, A. Richter and V.E.Verijenko, "Optimal design of symmetric angle-ply laminates for maximum buckling load with scatter in material properties", proceedings of AIAA Conference, (1994), pp. 1041-1045.

[17] D.Yadav and N.Verma, "Free vibration of composite circular cylindrical shells with random material properties Part I: general theory, Composite Structures, 41, (1997), pp 331-338.

[18] D.Yadav and N. Verma, "Free vibration of composite circular cylindrical shells with random material properties Part II: applications", Composite Structures, 51(4), (2001), pp 371-380.

[19] C.G.Bucher and C.E.Brenner, "Stochastic response of uncertain systems", Archive of Applied Mechanics, 62, (1992), pp. 507-516.

[20] K.H.Low, "A comprehensive approach for the eigen problem of beams with arbitrary boundary conditions", Computers and Structures, 39(6), 1991, pp. 671-678.

[21] V.S.Jagdish, "Static response of composite laminates with randomness in material properties and external loading", M.Tech. Thesis, Department of Aerospace Engineering, IITKanpur, India, May 1998.

[22] Amit kr. Onkar, "Nonlinear response of composite laminates with random material properties to random loading", M.Tech Thesis, Department of Aerospace Engineering, IITKanpur, India, June 2002.

[23] L.J.Gibson and M.F, Ashby, "Cellular Solids Structure and Properties", Pergamon Press, (1988).

[24] O.T.Thomsen, "High-order theory for the analysis of multi-layer plate assemblies and its application for the analysis of sandwich panels with terminating plies", Composite Structures, 50(2000) pp. 227-238.

[25] J.K. Palik, A.K. Thayamballi, G.S.Kim "The strength characteristics of aluminum honeycomb sandwich panels", Thin-walled Structures, 35(1999) pp.205-231.

[26] A.Muc, P.Zuchara, "Buckling and failure analysis of FRP faced sandwich plates", Composite Structures, 48(2000) pp. 145-150.

[27] W.Becker,"Closed-form analysis of thickness effect of regular honeycomb core material", Composite Structures, 48(2000) pp.67-70.

[28] D.E. Glass, V.V.Raman, V.S.Venkat, S.N.Sankaran, "Graphite/epoxy honeycomb core sandwich permeability under mechanical loads", Composite Structures, 44(1999) pp.253-261.

[29] V.Marimuthu, "Theoritical and FEM Free Vibration Analysis of Cylindrical Honeycomb Sandwich Laminate with Random material Properties", M.Tech Thesis, Department of Aerospace Engineering, IITKanpur, India, November 2001.

[30] J.N.Reddy and C.F.Liu, "A higher-order shear deformation theory of laminated elastic shells", Int.J.Engineering sciences, 23(3), (1985), pp.319-330.

[31] C.A.Shankara and N.G.R.Iyengar, "A c^0 element for the free vibration analysis of laminated composite plates", J.Sound and Vibration, 1914(5), (1996), pp.721-738.

[32] R.M.Jones, "Mechanics of composite Materials, McGraw-Hill, New York, 1958.

[33] J.N.Reddy, "Energy and variational methods in applied mechanics", John Wiley & Sons, New York, (1981).

[34] J.R.Vinson, "The behaviour of shells composed of isotropic and composite materials", Kluwer Academic Publishers, Boston, 1993.